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ABSTRACT

These papers from the fifth annual meeting of the Canadian Mathematics Education Study Group are organized similarly to the meeting, presenting materials from the lectures, working groups, special groups, and panel groups, as well as ad hoc groups. Iverson delivered one lecture on computer programming, stressing the place of APL, while Kilpatrick lectured on theoretical and methodological reasons why research in mathematics education has made little impact on classrooms so far. Working group reports concerned research and the classroom, computer education for teachers, issues in the teaching of calculus, and revitalizing mathematics in teacher education courses. Special groups reported on the place of geometry in the elementary school and an example of mathematics, the Rubik cube. Panels considered mathematics and language, and the relationship of history and the teaching of mathematics. In the appendix are reports from individual members of the working groups and panel groups, concerning a variety of mathematics education topics. (MNS)

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CANADIAN MATHEMATICS EDUCATION STUDY GROUP
GROUPE CANADIEN D'ETUDE EN DIDACTIQUE DES MATHEMATIQUES

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PROCEEDINGS OF THE 1981 ANNUAL MEETING

UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA

JUNE 5-9, 1981

EDITED BY DALE R. DROST

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CMESG/GCEDM

DECEMBER 1981

CMESG/GCEDM
1981 Meeting

PROCEEDINGS

Introduction iv

Editor's Forward v

Summary of the Conference 1

Lecture I Mathematics and Computers: APL as a Mathematical Notation - Kenneth E. Iverson 4

Lecture II The Reasonable Ineffectiveness of Research in Mathematics Education - Jeremy Kilpatrick 7

Working Groups:

A Research and the Classroom
Thomas E. Kieren, Shirley McNicol 37

B Computer Education for (Mathematics) Teachers
J. Dale Burnett 46

C Issues in the Teaching of Calculus
Ralph Staal 53

D Revitalising Mathematics in Teacher Education Courses
H.A.J. Allen 61

Special Groups:

P The Place of Geometry in the Elementary School
Dieter Lunkenbein 65

R An Example of Mathematisation: The Rubik Cube
Fernand Lemay 71

Panel Groups:

X Mathematics and Language
Martin Hoffman 82

Y Les Relations entre L'Histoire et la Didactique des Mathématiques - Bernard Hodgson 84

List of Participants 87

APPENDICES

A Contributions from Working Group A. 92

1. List of Participants 93
2. Jeremy Kilpatrick 94
3. Lars Jansson 95
4. David Wheeler 96
5. Walter Szetela 99
6. Louis Charbonneau 101

B Contributions from Working Group B. 102

1. Instructional Uses of Microcomputers
Marv Westrom 103
2. Copy of Newsletter LOGOPHILE 116

C Contributions from Working Group C. 121

1. Agenda Possibilities 122
2. Logarentials 126
3. Setting Up Integrals 127
4. By Parts 128
5. The Leibniz Notation - Hugh Thurston 130
6. Exponential and Logarithmic Functions
H.N. Gupta 146
7. Letter from R.R. Christian 152
8. Another Way to Introduce Natural Logarithms and e - R.R. Christian 156
9. Exerpt from Differential and Integral Calculus 161
10. A Problem - Hugh Thurston 163
11. Une Approache Fondamentaliste de l'Enseignement du Calcul Infinitesimal - Mario Lavoie, Yvan Roux 164

D Contributions from Working Group D. 174

1. List of Participants 175
2. Examples - Loïc Thérien 176
3. Du Nombre Pair a la "Preuve Par 9" - Renée Caron 183
4. An Example - Tom Bates 187
5. The Billiard Table - Keith Harborn 188

E Contributions from Panel Group X 189

1. Words Triggered by Images: Images Triggered by Words - A.J. Dawson 190
2. "X" Arguments for the Consideration of Language by Researchers in Mathematics Education
William Higginson 203

F Contributions from Panel Group Y 206

1. Contribution de Louis Charbonneau 207
2. Reading of Mathematical Classics Can Be Rewarding
H.N. Gupta 217
3. What Should a Teacher Know About the History of Mathematics? - David Wheeler 227

G Contributions from Production/Ad Hoc Groups 230

1. Mathematics in China - Tony Thompson 231
2. Le Dodécaedre et la Géométrie Projective d'Ordre 5 - Fernand LeMay 237

INTRODUCTION

These Proceedings are a partial record of the fifth annual meeting of the Canadian Mathematics Education Study Group. They are intended primarily as a reminder and a resource for those who participated in the meeting, but they may nevertheless contain material which will speak to people who were not there.

A lot is missing, of course, since the Study Group's work does not consist entirely, or even mainly, in listening to prepared presentations. Presentations are much easier to record and pass along than the lively, unpredictable discussions that take place, particularly in the Working Groups. Yet it is the latter that indicate that people are really working and generating new insights and new ideas. The writers of some sections of these Proceedings have tried to convey a sense of this mental activity, but inevitably with only very modest success.

The Study Group manages to hold together, better than some other collections of people who meet to talk about the teaching of mathematics, the essential strands of mathematics education. Whatever else it concerns itself with, mathematics education must attend to:

(i) psychological matters, such as the cognitive and affective aspects of human learning, thinking and problem solving;

(ii) mathematical matters, such as the content of mathematical knowledge, the historical and cultural aspects of mathematics, and the nature of mathematics as a human activity;

(iii) epistemological matters, such as how particular mathematical concepts and skills are generated and apprehended.

No doubt the Study Group has so far only managed to keep these three axes in view rather than integrate them into a solid structure, but it is certainly moving in a direction towards this difficult, and perhaps distant, goal. Any advance will strengthen discourse and research in the field of mathematics education immeasurably. Whether there will be any impact on the practices of mathematics teaching is more speculative. It is doubtful if anyone anywhere really knows how to bring about intended changes in our classrooms.

David Wheeler
Chairman: CMESG

EDITOR'S FORWARD

The organization of these proceedings reflects the organization of the meeting itself. The agenda included two lectures, four working groups, three special groups, and two panel groups, as well as the possibility of production/ad hoc groups. The proceedings are organized around the contributions of these groups.

The two lectures were given by Kenneth Iverson and Jeremy Kilpatrick. Dr. Kilpatrick's lecture is presented in its entirity, however Dr. Iverson's lecture consisted mainly of a demonstration of the use of the computer language APL via a computer terminal and was therefore not available as a paper. A brief comment by Dr. Iverson is included in the proceedings and the reader is directed to the references included with that comment for further information on his views.

Each of the various group leaders was asked to provide a short summary of his/her sessions and these are included in the appropriate sections of the proceedings. Reports were available for all of the groups with the exception of the special group on the art of solving and posing problems. In addition to the short summaries, the working group and panel group leaders submitted various contributions made by individual members of their groups. These contributions are not included in the body of the proceedings, however are included in the appendices.

Finally, two production/ad hoc groups were formed and the presentations are included in Appendix G.

Dale R. Drost
Editor

SUMMARY OF THE CONFERENCE

Canadian Mathematics Education Study Group
Groupe canadien d'étude en didactique des mathématiques

The fifth annual Meeting of the CMESG/GCEDM took place at the University of Alberta from June 5th to 9th, 1981. Approximately 50 people attended, most of them mathematicians and mathematics educators with positions in Canadian universites, and from every province except PEI.

The guest speakers this year were Dr. Kenneth Iverson (I.P. Sharp Associates), who challenged the Meeting with his view that existing computer languages are not mathematically equivalent alternatives, and that both mathematicians and educators will live to regret the pervasive effects of BASIC and other limited languages, and Dr. Jeremy Kilpatrick (University of Georgia), who examined the theoretical and methodological reasons why research in mathematics education has so far made little impact on classrooms. Other speakers were Dr. Murray Klamkin (University of Alberta), who shared some of his insights into the art of posing and solving problems, and Professor Fernand Lemay (Université Laval), who successfully undertook the unusual and difficult task of showing in an hour the stages of development of awareness in his conquest of Rubik's Cube.

Two panels of speakers introduced discussions of "Mathematics and language" and "The relation between the history and the pedagogy of mathematics". Groups discussed the teaching of geometry in elementary schools, the character of mathematics and education in China, looking ahead to ICME-V, and possible cooperation with the Science Council's project in science education.

This Meeting, following the precedent of other years, gave central importance to the Working Groups. The topics studied this year were (1) Mathematics education research and the classroom, (2) Computer education for teachers, (3) Issues in the teaching of calculus, and (4) Revitalising mathematics in teacher education programmes. Each member of the conference chose one Group and worked in it for a total of 9 hours. The size of the Groups ensures that everyone can actively participate, and the time alloted is enough for each Group to go beyond the obvious first stages in a topic and begin to work at the harder questions. There is no doubt that this feature contributes a great deal to the

generation of the atmosphere of serious and friendly cooperation which characterises these Meetings.

Anyone wishing further information about this or future Meetings may write to Joel Hillel, Department of Mathematics, Concordia University, 7141 Sherbrooke Street West, Montreal, Quebec H4B 1R6.

David Wheeler

LECTURE I

MATHEMATICS AND COMPUTERS

DR. KENNETH IVERSON

APL AS A MATHEMATICAL NOTATION

Kenneth E. Iverson
I.P. Sharp Associates
Toronto, Canada

The Speaker's remarks were based largely upon his Notation as a Tool for Thought [1], and are perhaps best summarized in the following excerpt from the introduction to that paper.

> The importance of nomenclature, notation, and language as tools for thought has long been recognized. In chemistry and in botany, for example, the establishment of systems of nomenclature by Lavoisier and Linnaeus did much to stimulate and to channel later investigation. Concerning language, George Boole in his Laws of Thought [1, p.24] asserted "That language is an instrument of human reason, and not merely a medium for the expression of thought, is a truth generally admitted."
>
> Mathematical notation provides perhaps the best-known and best-developed example of language used consciously as a tool of thought. Recognition of the important role of notation in mathematics is clear from the quotations from mathematicians given in Cajori's A History of Mathematical Notations [2, pp.332,331]. They are well worth reading in full, but the following excerpts suggest the tone:
>
> > By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race.
> >
> > A.N. Whitehead
>
> > The quantity of meaning compressed into small space by algebraic signs, is another circumstance that facilitates the reasonings we are accustomed to carry on by their aid.
> >
> > Charles Babbage
>
> Nevertheless, mathematical notation has serious deficiencies. In particular, it lacks universality, and must be interpreted differently according to the topic, according to the author, and even according to the immediate context. Programming languages,

because they were designed for the purpose of directing computers, offer important advantages as tools for thought. Not only are they universal (general-purpose), but they are also executable and unambiguous. Executability makes it possible to use computers to perform extensive experiments on ideas expressed in a programming language and the lack of ambiguity makes possible precise thought experiments. In other respects, however, most programming languages are decidedly inferior to mathematical notation and are little used as tools of thought in ways that would be considered significant by, say, an applied mathematician.

The thesis of the present paper is that the advantages of executability and universality found in programming languages can be effectively combined, in a single, coherent language, with the advantages offered by mathematical notation.

The cited paper is also available in A Source Book in APL [2], which includes other relevant material. Interested readers may wish to consult not only the cited paper, but "Algebra as a Language" (pages 35-45), the discussion of the mathematical roots of programming languages in the conclusion to "The Evolution of APL" (pages 70-74), the discussion of conventions governing the order of evaluation in mathematics (pages 29-34), and the discussion of the inductive method of teaching (pages 131-139).

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1. Iverson, K.E., "Notation as a Tool of Thought", CACM 23, August 1980, Copyright 1980, Association for Computing Machinery, Inc.

    1. Boole, G. "An Investigation of the Laws of Thought". Dover Publications, NY 1951. Originally published by Walton and Maberly, London, in 1954, and by MacMillan and Co., Cambridge.

    2. Cajori, F. "A History of Mathematical Notations", Volume II Open Court Publishing Co., La Salle, Illinois, 1929.

2. Falkoff, A.D. and Iverson, K.E., "A Source Book in APL", APL PRESS, Palo Alto, (Suite 201, 220 California Ave., Palo Alto, CA 94306), 1981.

LECTURE II

THE REASONABLE INEFFECTIVENESS OF RESEARCH IN MATHEMATICS EDUCATION

DR. JEREMY KILPATRICK

The Reasonable Ineffectiveness of Research in Mathematics Education

Jeremy Kilpatrick
University of Georgia

You may recognize the title above as a play on the title of an article by Richard W. Hamming, "The Unreasonable Effectiveness of Mathematics," which appeared in the American Mathematical Monthly in February 1980. In his article, Hamming asks the question, "Why is mathematics so unreasonably effective?" He offers some partial explanations but ends by saying that these explanations are so insufficient "as to leave the question essentially unanswered" (p. 82). I am going to follow the same strategy but with a different question. My question is, "Why is research in mathematics education so ineffective?" I shall offer some tentative thoughts on this question, but you will soon discern that I, too, must leave my question "essentially unanswered." In addressing my question, I shall make two claims: (1) much of the ineffectiveness of research in mathematics education is more perceived than real, and (2) most of the perceived ineffectiveness is reasonable.

The Effectiveness of Research in Education

Before asking whether research in mathematics education is really ineffective, I should like to broaden the context and ask whether research in education is, or has been, effective.

In 1978, the National Academy of Education published a thick

book--672 pages--edited by Suppes (1978), whose title suggests the positive claims that are made in its pages: Impact of Research on Education: Some Case Studies. The book contains nine studies that purport to show how educational research has contributed to school practice. Scriven (1980) has argued that the book does not address the "pay-offs" from educational research; it

> turns out to be . . . concerned with such questions as whether the results of basic research in psychology and the social sciences trickle down to the educational research journals (which, you will not be surprised to hear, they do). That does not show that educational research benefitted from that other research and, more importantly, it does not show that educational research benefitted anything else. (p. 10, italics in original)

Whether or not you agree with Scriven's analysis, the somewhat defensive tone Suppes adopts in the book's preface makes it clear that even in his view the effects of educational research have not been overwhelming:

> I like to think of the case studies presented here as representing only a sampling of the beginning of educational research in its early years of development. What the future holds should be brighter and better, because it will be able to build on the kind of work reported in this volume and, as that building takes place, the impact on practice should become more marked. (p. xvi)

Some of the difference of opinion between Scriven and Suppes

concerns whether one should talk about effects (impact) or benefits. Scriven is pushing a much more demanding criterion than is Suppes. But even if one limits the question to effectiveness, one has only to glance at the pages of journals such as the Educational Researcher over the past couple of years or attend some of the sessions at the annual meetings of the American Educational Research Association to sense that there is a crisis of faith in educational research. Have we been doing the wrong things? Have we failed to make contact with school practice? Who, if anyone, is listening to what we have to say? Is it all an empty exercise? These are the kinds of questions one senses are below the surface of the articles and talks. Many of these same questions undoubtedly occur to researchers in mathematics education, but we do not seem to be addressing them in even the oblique fashion of our colleagues in educational psychology.

The issue of whether or not educational research is effective seems to get tangled up, in some people's minds, with the distinction between pure and applied research. What is the difference between pure research and applied research in education? Which is more likely to have a greater impact on educational practice? Two models seem to be implicit when people make the distinction between pure and applied research.

The first model is hierarchical. Basic research is at the top--so high in status that it is often seen as "up in the clouds." Below it is applied research, and below that somewhere is the mundane world of practical affairs. The simplest version of this model is what Greeno (1978) calls "the pipe-line model":

> According to this model, fundamental knowledge and theories are like crude oil, which gets pumped out of the ground in basic research. Basic knowledge is shipped to applied research settings where it is transformed into something more useful; this is like shipping crude oil to refineries, and transforming the product into useable forms. Finally, results of applied research are shipped to developers and disseminators who use the knowledge in making products for use in school and send the stuff around to school users. This is analogous to sending refined gasoline to filling stations, where customers can drive up and fill their tanks. (pp. 7-8)

Greeno argues that this model is at best a weak reflection of the relation between science and technology in any field, and that it is grossly misleading in education. He argues that basic and applied research are not hierarchical; they overlap substantially, each contributing to the other, and the significant research questions are both basic and applied. I shall return to this argument after further consideration of the hierarchical model.

People who do not adhere to the oversimplified pipeline model may still view basic research as higher in status because it is theory-oriented and because it aims at generalization, whereas applied research, with its lack of theory and specificity, is necessarily lower in status. Suppes (1967), in an article written for a booklet that was the forerunner of the Journal for Research in Mathematics Education, seemed to adopt this hierarchical view. He argued that basic research could have a direct impact on practice, that we needed more basic research in mathematics education, and that we needed both

theoretical and empirical basic research. He seemed to be suggesting that basic research in our field should be largely concerned with how students learn mathematics, that applied problems in our field include more effective ways of organizing the curriculum, and that basic research will necessarily be more helpful than applied research in addressing such curriculum questions. I should not have to point out that this is a somewhat restricted view.

Tyler (1981) seemed to assume the same sort of hierarchy between basic and applied research recently when he spoke to some directors of projects sponsored by the National Science Foundation. Where he uses "science education," one can read "mathematics education":

> Research needed in science education is not only basic research which results in widely generalizable concepts and principles but also applied research, that is, inquiries focused on particular situations and particular kinds of students, teachers, and institutions, which furnishes information of importance in improving science education in the particular circumstances where efforts to improve science education are being made. (p. 6)

Again, one has the connection between basic/general and applied/specific and, it seems to me, an implied status differential.

The other dominant model for thinking about basic and applied research might be called the <u>complementarity</u> model: the two types of research are seen as complementary, each with its own domain and its own agenda, and equal (supposedly) in status. When one sees basic research characterized as "conclusion-oriented" and applied research

as "decision-oriented," one can be fairly sure these types of research are being viewed as separate but equally valid--for their own domain of relevance. Other ways to assert their complementarity are to characterize basic research as descriptive and applied research as prescriptive, or basic research as being concerned with learning and applied research with teaching.

Adherents of the complementarity view (Gelbach, 1979; Greeno, 1978) seem agreed that the distinction between the two types of research has become increasingly blurred in recent years. Gelbach argues that "it is now almost impossible to discriminate" (p. 9) between basic and applied research because of: (1) new capabilities in research methodology (such as multivariate statistical methods) and design that permit the investigation of practical instructional problems in natural classroom settings with the same "scientifically respectable levels of precision" (p. 9) that one has in investigating basic research problems; and (2) important arguments recently advanced by people such as Glaser, who claim that our theory-building should be prescriptive, not descriptive. Snow has argued that we should abandon attempts at general theory construction in favor of less ambitious, but more achievable, "local" theories; for example, "theories that apply to the teaching of arithmetic in grades 1-2-3 in Washington and Lincoln schools in Little City, but perhaps not to the two other elementary schools in that town" (Snow, quoted by Gelbach, 1979, p. 9). Gelbach criticizes this view, saying that "local theory development should be our <u>last resort</u> rather than our next move" (p. 9, italics in original). Certainly, Snow's position implies a convergence between basic and applied research.

I propose a third model for the difference between basic and app-

lied research that can be used to address the question of the effectiveness of research. Although it may appear to conflict with the other two models, I think it is equally valid. Recall the story of the married couple who went to the rabbi to help resolve their quarrel. The wife went in to the rabbi and told her story, whereupon the rabbi nodded and said, "You are right." Then the husband told his side of the story to the rabbi, who nodded and said, "You are right." After the couple had gone on their way, the rabbi's wife, who had heard both exchanges, said to her husband, "They cannot both be right." And the rabbi nodded and said, "You are right." It is in this sense that I claim my model is right.

When people talk about research as being "basic" or "applied," "conclusion-oriented" or "decision-oriented," "descriptive" or "prescriptive," and so forth, they are referring not to anything that can be considered an intrinsic quality of the research study itself, but rather to either the researcher's purpose in conducting the study or the uses to which the study is put. In other words, the same study can be either basic or applied, depending on who is doing the labeling and for what purpose. Basic research is not defined by whether it is conducted in a laboratory rather than a school, nor is it defined by whether analysis of variance is used rather than chi-square. One cannot unambiguously label a piece of research as either basic or applied; one can only ask what connection the research study appears to have to theory and what connection to practice. Both of these, to a large extent, are in the eye of the beholder.

Consider a hypothetical situation: a researcher conducts a study and then writes a report. Her purpose may have been to understand and explain some phenomenon that has to do with the learning of

mathematics. She may hope to derive some generalization from the results. Her purpose is basic research--conclusion-oriented research. That is her purpose, but that does not mean the study was basic research in some intrinsic sense. We are speaking only of her purpose. When the report is written, of course, her view will be woven, more or less explicitly, into the account. A reader of her report brings his own frame of reference to it. Although it is important to distinguish between a research study and the report of the study, especially when writing about one's own work, there is a sense in which, for the reader, the study <u>is</u> the report. The reader ordinarily has no firsthand knowledge of a study other than what is contained in the report. More precisely, for the reader, the study consists of the report plus the frame of reference in which he embeds it. A given reader may see our researcher's study as applied research, despite her avowed purpose. She may have failed to make a clear link with theory, even though she intended to do so. The reader may have a practical problem to solve and may be able to use the results of the study to help solve it.

With respect to a particular piece of research, which can only be known to a public through some report, the classification of basic versus applied depends upon the perspective of the reader of the report. This model might be called a <u>lens</u> model--a study may be basic or applied depending upon the lens you use in reading a report of it. This lens ought to be understood as incorporating your purposes and intentions in extracting information from the report. If the report helps you formulate a theory, then the study is functioning as basic research, regardless of the author's intentions. If the study helps you solve a practical problem, then the study, for you, has been app-

lied research.

This point of view has implications for the effectiveness of educational research in general and research in mathematics education in particular. From this point of view, effectiveness, too, is relative. It is relative to both the criterion used and the person using the criterion. Consider the argument presented by Getzels (1978) for the effectiveness of basic research in education. He characterizes basic research as follows:

> Basic research comprises studies in which the investigator formulates his own problem regarding a phenomenon or issue, and his aim is primarily to conceptualize and understand the chosen phenomenon or issue and only secondarily, if at all, to do anything about it. The work is theory oriented rather than action oriented. Although the distinction here is not foolproof, in this view basic research (like fine art) deals with "discovered problems" and applied research (like commercial art) deals with "presented problems." (p. 480)

This is a nice formulation, but it hinges on the "aim," the intention, of the investigator. It characterizes the research study as seen by the investigator, but not necessarily as seen by others. Getzels continues as follows:

> Despite the belief that basic or theory-oriented research has little effect on practice--a belief on which the sacrifice of basic research for other activities is founded--the fact is that basic research can have powerful effects on practice. (p. 480)

Getzels then gives several specific examples of such effects.

How is it that someone like Getzels can so confidently assert that basic research can have powerful effects on practice, when our own experience as practitioners has suggested to most of us that such effects are rare, if not unknown? Is it a difference between mathematics education and the rest of education? Or is it a difference attributable to our perspective, as opposed to Getzels'? Here, too, I shall argue that both are right.

The Ineffectiveness of Research in Mathematics Education

Let us turn to research in mathematics education and ask why it appears, to many, to be ineffective. One reason may be that, despite what appears to be a flood of research in our field, we actually have very little in the way of research to go on in drawing implications for practice. As many observers (e.g., Greeno, 1978) have noted, the amount of money spent on research in education by the federal government in the United States is a small fraction of that spent on other areas of research such as national defense, space research, or atomic energy. Moreover, the amount spent on research in education compared to the amount spent on education in general is an even smaller fraction; one estimate is that it is less than 0.4% (Getzels, 1978, p. 478). Mathematics education, although it has done comparatively well, has shared in this dearth of funding. Again, it is a matter of perspective. Viewed one way, a lot of money has been spent by the US federal government over the last decade or so to support research in mathematics education. Viewed another way, however,

the funds have not been nearly enough to do the job properly.

We face another paradox, too. It sometimes seems as though, amid the frantic activity of research in mathematics education, we must have more than enough data to answer important questions that face us. Begle (1979) surely had this impression as he undertook the massive research synthesis that resulted in his book, Critical Variables in Mathematics Education. Yet one can convincingly argue that we do not have enough data--certainly not enough of the right sort of data. As Bauersfeld (1979) noted: "We have a shortage in the midst of abundance" (p. 210). Sanders (1981), speaking of educational research as a whole, recently argued that we lack

> a body of systematically observed, factual knowledge about the way education operates in its natural settings. . . . Despite rampant empiricism, there is no extensive "figuration of facts" . . . observed regularities of the empirical world which must be accounted for by scientific explanations. Although we have large quantities of census-type data, achievement and other test data, there is very little trustworthy data representing the facts of the educating process. (p. 9)

Sanders calls for more case studies and naturalistic investigations "to redress this weakness" (p. 9). His argument applies as well to research in mathematics education as to educational research in general.

Another partial explanation for the apparent ineffectiveness of research in mathematics education has to do with our lack of what

Bauersfeld (1979) termed our "self-concept." Researchers in mathematics education do not constitute a true community. In North America, we have the Canadian Mathematics Education Study Group, the Special Interest Group for Research in Mathematics Education, the North American branch of the International Group for the Psychology of Mathematics Education, the Research Council for Diagnostic and Prescriptive Mathematics, the informal networks spawned by the Georgia Center for the Study of Learning and Teaching Mathematics, the research sessions at the meetings of the National Council of Teachers of Mathematics, the Journal for Research in Mathematics Education, and so on. Despite all of these activities--and perhaps because of all of them--we lack a strong common identity; we are not truly a community.

You have undoubtedly heard the refrain that most of the research studies in our field are conducted as part of the requirement for a doctorate and that most of these are done by people who will never do another piece of research. It is an old refrain, but unfortunately it seems to be as true today as it ever was. The annual surveys of research that have been published in the Journal for Research in Mathematics Education during the last decade show that, although the growth in the number of dissertations cited may have lagged a little behind the growth in the number of journal articles, there are still something like two dissertations for every article. Further, it appears that the overwhelming majority of the dissertations do not come from departments of mathematics education, nor are they conducted under the supervision of people who are recognized researchers in mathematics education. They may be good dissertations, and they may come from worthy programs in worthy institutions. But they do not arise from what one might call "the research community in mathematics

education." They do not partake of issues that concern this community; they do not arise from common concerns, shared knowledge, mutual interaction. Is it any wonder that collectively they do not add up to very much?

Insufficient funds, insufficient support, insufficient knowledge, insufficient collegiality--are these not good reasons for the perception of research in mathematics education as ineffective? Is it not reasonable that research conducted under such conditions would fail to influence school practice? There are two additional reasons, however, that appear most compelling of all: (1) our lack of attention to theory, and (2) our failure to involve teachers as participants in our research.

Attention to Theory

I recently examined the 35 (out of 38) articles in the ten issues of the Journal for Research in Mathematics Education from July 1979 to May 1981 whose authors had affiliations with US institutions only. I looked at each article to see if an attempt had been made to link the question under investigation to some theoretical context. For 20 of the articles, I could find no such attempt. I may have been too harsh in some of my judgments, and I may have been somewhat hasty, but this and other observations convince me that a lack of attention to theory is characteristic of US research in mathematics education. This conclusion may not apply to research done in some other countries, but the problem is not unique to the United States.

Why is this lack of attention to theory such a serious problem? I contend that it is only through a theoretical context that empirical research procedures and findings can be applied. Each empirical

research study in mathematics education deals with a unique, limited, multi-dimensional situation, and any attempt to link the situation considered in the study with one's own "practical" situation requires an act of extrapolation. Extrapolation requires, however, that one embed the two situations in a common theoretical framework so that one can judge their similarity in various respects. As the old adage has it, "There is nothing so practical as a good theory." Kerlinger (1977) has argued that "the basic purpose of scientific research is theory" (p. 5) and that "there is little direct connection between research and educational practice" (p. 5). The effect of research on educational practice is indirect; it is mediated through theory. As Kerlinger points out, two factors that in the long run hinder the effectiveness of educational research are the twin demands for payoff and relevance. Such demands short-circuit the theory-building process.

Let us consider some examples of how theory has, or has not, affected practice in mathematics education. A frequently cited example (Cronbach & Suppes, 1969; Resnick & Ford, 1981) is E. L. Thorndike's influence on the teaching of arithmetic during the early years of this century. There is no doubt that Thorndike, through his research, his teaching, and, most especially, his analysis of the psychology of arithmetic, substantially influenced the teaching of school arithmetic in the United States. He was one of the few educational theorists to be actively concerned with the nuts and bolts of curriculum building. His theoretical ideas had an impact in the classroom largely because he himself (and his students) analyzed textbooks in the light of his theory and made concrete suggestions for changes. His theory was his hammer; he looked around and saw the arithmetic curriculum as

something to pound. One should perhaps note that he did not have much competition at the time and that he was extremely energetic in his efforts to apply his theoretical ideas. He was not, strictly speaking, a mathematics educator, and his research, strictly speaking, was not research in mathematics education, but we put it there quite happily. He is a notable exception to the charge that researchers do not influence practice in our field.

A second example is Piaget, also--needless to say--not a mathematics educator. Groen (1978) has assessed the impact of Piaget's theoretical ideas on educational practice, and he devotes one section of his assessment to mathematics. Groen begins by noting that "the hard core of Piagetian theory is replete with mathematical analogies" (p. 299), and consequently, "it is not surprising that there are many parallels between mathematics education and Piaget's own ideas" (p. 299). Groen contends that, usually, rather than Piaget influencing the teaching of mathematics, it was the other way round--mathematics influenced Piaget's thinking. Groen then raises the issue of discovery learning and argues--with considerable justification--that on this issue the influential theorist has been not Piaget, but Polya. Further, he argues that with respect to "the notions of mathematical competence underlying the 'new math' curricula" (p. 300), the applied research done under the Piagetian influence dealt with highly specific problems and was difficult to generalize from. Groen concludes with an analysis of Copeland's book for elementary school teachers on the teaching of mathematics. He claims Copeland gives a one-sided view of Piagetian theory that emphasizes its static aspects and that tends to confuse mathematical structure with Piaget's more dynamic view of structure.

One might reasonably conclude from Groen's assessment that Piaget's ideas had not had much influence upon mathematics teachers. A more valid conclusion is that Piaget's ideas, as the teachers understand them, have had a profound impact, but this impact is often difficult to discern clearly. In countless classrooms today, mathematics teachers are dealing with children and teaching their subject matter in the light of what they believe to be Piaget's ideas. It is part of the professional baggage they picked up in college that is still with them, and it is heavily reinforced by the professional culture in which they live. Although Groen apparently could not find much of an overt Piagetian influence on mathematics education, the influence has been substantial, but largely covert and indirect.

Let us consider a final example of the influence of theory on practice in mathematics education. Several years ago, Stake and Easley directed a series of case studies of science and mathematics teaching for the National Science Foundation (see Fey, 1979). They found a number of secondary school mathematics teachers who offered, as justification for teaching their subject, the argument that the study of mathematics improves one's ability to think logically:

> "I can teach them to think logically about real problems in their lives today."
>
> "Mathematics can teach the student how to think logically and that process can carry over to anything. To be able to start with a set of facts and reason through to a conclusion is a powerful skill to have." (quoted in Fey, 1979, p. 498)

These teachers had clearly rejected Thorndike's findings concerning

the lack of transfer of the disciplines--if indeed they had ever heard of these findings--and had adopted a view that has echoes of faculty psychology. Presumably this view was not dominant in their preservice education program, which doubtless gave them much sounder, more scientific justifications for the teaching of mathematics. These justifications either had not survived or had never been accepted. The educational psychology textbooks are fairly clear on this issue: one cannot train logical reasoning ability through specific school subjects like mathematics. This is part of the received wisdom of the school-of-education culture, and these teachers must have been taught it. We have here a case in which current theories have not had much impact on teachers' thinking, and presumably their practice.

These three examples are intended to illustrate some of the various and perhaps perverse ways in which theory influences practice in mathematics education. As Kerlinger and others have noted, the influence is primarily indirect. Unless someone forceful and dominant such as Thorndike acts on the system, one must look hard to detect how the influence is occurring. A common procedure is for the theorist to set forth his views and then for a transmitter, such as Copeland, to provide a simplified, and perhaps somewhat garbled, version for a larger public of teachers. The transmission network, however, is complex. A Piaget introduces a new idea, which resonates for someone else, who incorporates it into a talk, paper, or book, and other mathematics educators begin to use it in their speaking or writing. Gradually, the idea comes into the culture of mathematics education and is picked up by teachers in practice. Sometimes the idea is banned from colleges of education--like faculty psychology--but lurks in the culture like a virus to strike down the receptive practitioner.

Sometimes the force of theory is felt merely by providing a name for a construct that people have been grappling with but have not articulated. Attribution theory and expectation theory seem thus far to have contributed to mathematics education in this fashion; researchers in mathematics education are intrigued by the constructs, but they have not been much concerned with following out the ramifications of the theories. Naming, however, is a powerful force, as Adam must have discovered. Hadamard (1947), in discussing Newton's contributions to the calculus, said it aptly:

> The creation of a word or a notation for a class of ideas may be, and often is, a scientific fact of very great importance, because it means connecting these ideas together in our subsequent thought. (p. 38)

Pimm (1981, p. 48) quotes Higginson's anagram, "re-nameing is re-meaning," from which it follows that "nam(e)ing is meaning." We need the constructs and networks of theory to help us think about things--about the phenomena we confront as mathematics educators. We ought to be giving more serious attention to the theoretical underpinnings of our work, and we need to make more explicit and coherent the assumptions we are making, the point of view we are adopting, and the frame of reference that surrounds the picture we are trying to paint. As long as we ignore the theoretical contexts of our research work in mathematics education, it will remain lifeless and ineffective.

Teachers as Participants in Research

Consider now the teacher's role in research. First, we should quickly note that "research" should be given the broadest possible

connotation; we should not limit it to controlled experimentation or even to empirical research, as is often done. Research in mathematics education should include historical studies, philosophical studies, and analyses of curriculum topics, as well as surveys, case studies, clinical studies, and the like. What makes a study research is not the methodology but the attempt to be systematic and to put the study in a larger context of theory, if possible. (This is true even for what some would term "applied research.") "Disciplined inquiry" is perhaps a better term in some respects than "research" since it emphasizes the process and not just the form.

From this perspective, one can see that much of what mathematics teachers do every day comes close to being research; it is just not quite so deliberate, systematic, or reflective. As Alan Bishop (1977) has pointed out, teachers can borrow three things from researchers: their procedures, their data, and their constructs. What do researchers do when they do research? If they are conducting an empirical study, they might observe, formulate hypotheses, observe some more, and try to test their hypotheses. If possible, they try to vary the situation systematically to see what the effects of variation might be. They formulate constructs and models of how these constructs might be related, and then they gather data to test the constructs and models. They develop instruments to help them gather data. These are activities that teachers can do. They can borrow these procedures and use them to study their own teaching. They can also borrow researchers' data. As Bishop points out, you do not have to gather data yourself for them to be of value to you. The value of data is in the process of understanding and interpreting them. Teachers can interpret the data from a research study in the light of

their own situation and experience.

Teachers can also borrow a researcher's constructs and the accompanying models and theories. Bishop refers to the work of George A. Kelly, the developer of the psychology of personal constructs, which is a theory of personality functioning. In Kelly's theory, we are all researchers, creating constructs as interpretations of our world and testing the predictive validity of these constructs. When we teach, we are concerned with the students we are teaching and with the ideas we are trying to teach them. Our behavior is shaped by the constructs we have about the students and the ideas. The students, in turn, have their own constructs about us and about the ideas as they understand them. Kelly sees behavior as an experiment. To understand a child's behavior, says Kelly, try to figure out what question she is asking of the world. What hypothesis is she attempting to test? To change the child's behavior, try to figure out ways of getting her to form new constructs. To change one's own behavior as a teacher, try to create alternative constructs for interpreting the world. If you cannot create such constructs, try borrowing some. The great value of the work of theorists such as Piaget, Dienes, and Gagne is in the interpretive lenses they give us for looking at familiar phenomena in new ways.

Too many mathematics educators have the wrong idea about research. They give most of their attention to the results. They think it is primarily important for teachers to know the results of the research on a given topic. They give a high priority to summarizing and disseminating research results so that teachers can understand them. In a nontrivial sense, however, the results are the least important aspect of a research study. Note that Bishop did not

include results among the things to be borrowed from researchers.

The most important aspect of a research study is the constructs and theories used to interpret the data. A landmark research study is one that confronts us with data analyzed and organized so as to shake our preconceptions and force us to consider new conceptions. A researcher makes a contribution to our field by providing us with alternative constructs to work with that illuminate our world in a new way, and not simply by piling up a mass of data and results.

This view suggests why teachers should be active researchers, why they should develop a research attitude. Teachers should not stop at being borrowers; they should become collaborators. Research is not something to be left to people who understand randomized block designs and analysis of covariance. Research in our field is disciplined inquiry directed at mathematics teaching and learning. It is stepping out of the stream of daily classroom experience and stopping to reflect on it. It is becoming conscious of the constructs we are using and then trying other constructs on for size.

Research in mathematics education has increasingly been moving out into the classroom. This has been, in general, a healthy move. It would be better, however, if teachers were working more closely with researchers in formulating their problems and interpreting their findings and not simply in helping them gather data. The teachers would benefit, with respect to both their professional attitudes and their effectiveness, and so would the researchers.

Sanders (1981) has suggested that in no other profession is the community of researchers more sharply differentiated from the community of practitioners than in education. Researchers tend to identify with, and publish for, communities that do not include

practitioners, and vice versa. The self-correcting mechanisms of science, however, require that the knowledge it claims is reliable be presented to a community of peers for review and correction. If incorrect or incomplete theories become institutionalized, asks Sanders, where will the impulse for correcting them come from? At present, the theory builders in our field, such as they are, do not see the consequences of their ideas in practice; and the teachers, who have been trained to depend on experts for answers, have little impetus to correct these ideas and improve their own understanding.

Certainly the interests of the teacher and the researcher are not necessarily congruent. Neither one should expect too much from the other (Phillips, 1980), but this by no means invalidates the argument that each can profit from a closer association with the other.

A Contrast between Mathematics and Research in Mathematics Education

Hamming (1980) offers four partial explanations for the effectiveness of mathematics that may help to explain further the ineffectiveness of research in mathematics education. Hamming argues, first, that we see what we look for; "we approach . . . situations with an intellectual apparatus so that we can only find what we do in many cases" (pp. 88-89). The phenomena we see arise from the tools we use, and mathematics has been highly creative in inventing tools. Research in mathematics education, on the other hand, has not. Hamming relates a parable he attributes to Eddington: Some men went fishing in the sea with a net and, upon examining what they caught, found that there was a minimum size to the fish in the sea. In research in mathematics education, our nets have been rather coarse; our instruments, rather blunt.

Second, we select the kind of mathematics to use, and "it is simply not true that the same mathematics works every place" (p. 89). When the mathematics we have does not work, we invent something new. Hamming gives the illustration of how, when scalars did not work for representing forces, vectors were invented, followed by tensors. In research in mathematics education, we have not shown the same ingenuity in adapting our tools to our problems.

Third, science in fact answers comparatively few questions. "When you consider how much science has not answered then you see that our successes are not so impressive as they might otherwise appear" (p. 89). If one considers the questions associated with truth, beauty, or justice that mathematics cannot answer, says Hamming, one sees that almost none of our experiences fall under the domain of mathematics. Applying this same argument to the realm of research in mathematics education, one concludes that perhaps mathematics educators have not recognized the limits in the classroom to the kinds of questions that research might be able to answer. Perhaps one reason for the perceived ineffectiveness of research in mathematics education is that too much has been expected of it.

Fourth, Hamming contends that the evolution of man has provided the model for mathematics by selecting for "the ability to create and follow long chains of close reasoning" (p. 89). We have been, to some extent, selected according to the models of reality in our minds. For example, we think very well about problems pertaining to things that are about our size, says Hamming, but we tend to have trouble if the problems concern very large or very small things. Just as there are some light waves we cannot see and some sounds we cannot hear, perhaps there are some thoughts we cannot think. Although evolution has not

had much chance to operate over the few generations of scientists in the history of science, perhaps there has been some selection for the ability to follow chains of reasoning. The history of research in mathematics education is much shorter, and evolution has not had time to select for a research attitude. Many people who do research in mathematics education have, in fact, been "selected"--or have selected themselves--because of their mathematical abilities. These abilities may not be, and probably are not, the same abilities that are needed for effective research in mathematics education.

Davis and Hersh (1981) argue that the Platonist, formalist, and constructivist views of mathematics are no more than different ways of looking at the same thing. They use the analogy of how one can sit at the console of an interactive graphics system and learn about a hypercube by looking at pictures of the hypercube, rotating it so as to see how one view transforms into another. The viewer gradually builds up a comprehensive view of the thing itself out of the various partial views displayed. Similarly, one can build up a picture of mathematics itself by integrating the various pictures of it that are offered by the various philosophies of mathematics. Research in mathematics education may also be something like the hypercube, except that we are just beginning to note various views of it. Partial views are offered in several recent sources such as Begle (1979) and Shumway (1980), but a comprehensive image remains elusive.

The parallel between mathematics and research in mathematics education ought not to be pushed too far, however. Applying educational research to mathematics teaching practice is not an engineering problem like applying mathematics to a practical situation. For too long researchers have been misled by this

engineering metaphor. The improvement of mathematics teaching is not a technological problem; it is a human problem. Kristol (1973), writing about the inability of the reforms of the 1960s to have much impact on the educational process, put it this way:

> There are some who will say that this state of affairs merely shows how obstinately conservative our "educational establishment" is. I think this misses the point. When there is so much will to change, so much dedication to effecting change, and so little effectual change, the more reasonable conclusion is that we are dealing with a network of human relationships that does satisfy, if only in a minimal way, certain basic societal needs, even if we don't quite know why or how it does. . . . That this should surprise us indicates how deeply our thinking about all subjects has been suffused with the technological mystique. We are inclined to believe that our power over nature and humanity is, or ought to be, limitless. We tend to assume that the will to transform our human condition is a sufficient condition for such a transformation to occur. Everywhere, we hear the refrain: "We can go to the moon, can't we? Well, why can't we do something equally marvelous about the ghettos or education or whatever?"
>
> The answer is, of course, that going to the moon

> is easy whereas improving our system of education is hard. The one is nothing but a technological problem, the other is everything but a technological problem. Doing something about education means doing something about people--teachers, students, parents, politicians--and people are just not that manipulable. They are what they are and do not become new people to suit any new ideas we might have. (p. 62)

If researchers in mathematics education are to become effective in improving the practice of mathematics teaching, they should: (1) develop a stronger sense of community, which would include practicing teachers as collaborators in research; (2) create their own theoretical constructs for viewing their work; and (3) recognize the limits of their domain as well as its complexity.

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WORKING GROUP A

RESEARCH AND THE CLASSROOM

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RESEARCH AND THE CLASSROOM

A Report of the CMESG/GCEDM Research Working People

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The title of the group suggests a number of possible connotations:

- what is true both in research and practice?
- what is the bridge(s) from research to practice or vice versa?
- what are implications of research for practice or vice versa?
- *what are ways in which research interests can be projected to the classroom?

It is this latter question, inspired by a presentation at the 1979 CMESG/GCEDM meeting by Jack Easley, which provided a focus for the group.

Because of the nature of the group, some 19 university persons interested and active in mathematics education research, such a question is both appropriate and well within the scope of such persons' responsibility and perview.

The indicated question above leads to a variety of sub-questions examples of which are:

a) what are "spin-offs" from research useful in classrooms?
b) how can teachers be involved and interested in research?
c) how might research work effect various aspects of teacher education?
d) what are the researcher's responsibilities in relating his/her work to practice?

All of these questions are obviously related to the general topic

and like it have "direct" and "indirect" components. That is, in each case one can ask "is the impact or projection directly on to an aspect of practice or indirect, requiring further interpretation and elaboration?

In its deliberations, the working group focussed mainly on the indirect aspects of questions a,b, and c above. There were a considerable number of comments on question d; this will be addressed later. The working style of the group was to address these questions by looking at five examples of recent Canadian mathematics education research, each presented by one of the group members who had worked on it.

Kieren gave the example of the Rational Number Thinking Test as a research tool which teachers might use. This test is based on previous work by Gerald Noelting as well as Kieren, Nelson and Southwell, (which had been discussed at previous CMESG/GCEDM meetings in 1978, 1979 and 1980). The test contains four parts based on different mathematical interpretations of rational numbers. Within parts the questions or tasks represent a heirarchy of thinking. Research evidence based on use with over 1500 children and young adults indicates that the test might allow a teacher to observe the thinking tools that a student uses with respect to rational numbers, to see her/his differential reaction to various mathematical situations, and to see the interaction of language use and problem setting. In addition, according to Brindley's work in Calgary, the test can be used to classify students as concrete, transitional or formal with respect to rational numbers.

The group's discussion of this "spin-off" of research for practice revolved around three points. The first was how this test could be used in classrooms. The second was should it be used. That is, is such a test sufficiently related to a teacher's or a student's work to prove useful. Third, what messages from research are contained in such a "spin-off" test. With regard to this latter question, Gaulin made several stimulating observations. The first was that the test is in itself a reflection of both theoretical and empirical research and in that sense carried research ideas to the teacher. Second, the test above, with documentation of uses and results could inform practice in an indirect way. Third, the content and structure of the test contained a "hidden message" about how the curriculum for rational numbers might be organized. Thus an instrument or curriculum piece which comes from research and is used in practice might represent a projection at three levels:

- the theory behind the material is carried by the material itself,
- the results of research use could be projected to further use in the classroom,
- the material might be prototypic for other curriculum ideas.

Harrison's presentation of the Calgary Junior Mathematics Project focussed the group's attention on how teachers can stimulate and participate in research. This project had its theoretical underpinnings in the work of Piaget, Bruner and Skemp and the curriculum research of Bell. Harrison had a research group of teachers who both studied prepared process oriented materials and developed process oriented techniques which they would use in the research in their classrooms.

Following his carefully prepared summary of the development and implementation of the Project, discussion centred on the ways in which this research can be projected in the classroom. The group concurred with Harrison's response that the Project allowed for a) teacher involvement in the orientation and workshop sessions (where over 200 teachers shared materials), b) spin-off benefits to school colleagues and pupils, c) the dissemination of results at conferences in Berkeley, St-Louis and Grenoble.

The third example of research considered by the group was that of David Wheeler on problem solving, specifically "an investigation of the mental operations of high school students in solving mathematical problems". In his report on the uncompleted Project, Wheeler stressed:

a) difficulties involved in identifying "mental operations" - preferring instead to use the term "strategies", though with less satisfying results.

b) difficulties in identifying and analysing appropriate protocols & preferring to use "episodes" rather than each spoken contribution.

c) benefits of the "clinical interview" techniques especially when the interviewer takes a positive role in the situation.

In the brief discussion which followed, the group attempted to answer the question of how this research can contribute to mathematics teaching in the classroom. In addition to the obvious value to the teachers involved, as evidenced by their surprise at students' responses, it was generally agreed that this type of research is of greater interest to the constituency of researchers. (A more detailed account of Wheeler's Project is included in Appendix A).

The fourth research example took the form of video-taped interviews of children relating number ideas to a series of geometric tasks. This research done by Lunkenbein and presented by him was considered by the group for its implications for teacher education as well as implications for elementary school curriculum. The video tape itself shows a way in which teachers might use research. In a classroom setting a teacher rarely has the chance to observe individual students in a detailed or elaborate way. Such a video tape represents a controlled way for a teacher to get an opportunity to observe children doing mathematics. This control stems from the fact that the mathematics, the task, and the protocol have been predetermined by the researcher. The teacher should be able to find out the researcher's assumptions, but then observe the child for their own purposes; that is the teacher can predict behaviors of children and then test those ideas through observations. The group also saw such video tapes and related materials as a way of injecting a research objective or a sampling of functioning in a research made into the teacher education curriculum. Use of tapes would allow a person to get an image of the theory of the researcher, and to observe the materials and techniques which could be modified for informal use in the classroom.

Gaulin presented a longitudinal study of the use of calculators in upper elementary school mathematics. The group's discussion touched on a wide variety of questions related to research and the classroom. Gaulin continued to remind the group of the many audiences for research projections or implications.

Thus, for example research might have a "political" implication for provincial education authorities. In the case of the research example, research results comparing calculator and non-calculator groups on a number of dimensions of achievement. Thus, while the researcher might be interested in the more phenomenological aspects of calculator use, he or she reasonably may be obligated to collect information for particular constituents or funding bodies.

In the area of "spin-off", Gaulin reported that teachers were very interested in the collection of problems for calculator solution. While these were used in research to provide children with potentially rich settings for calculator use, the teachers saw them as useful in the curriculum in a more general way.

Conclusions

As a group, a main conclusion from our work was that we did not get very far in an effort to relate research and practice. Yet a number of useful observations did arise out of our sessions:

1. Spin-off materials tests, curriculum pieces, techniques - bear (or should) information about research and theory for their users.
2. Most research materials contain messages about potential change in the curriculum or in the actions of teachers or children. These could be made more explicit than is currently done.
3. Teachers involved as research colleagues help define research in "practical" terms. Such teacher/researchers can communicate the particular research in which they are involved to others as well as helping develop a research "face" in other teachers.

4. The fact that much current research involves detailed observations of children and young adults in mathematical settings, can be used to great advantage in teacher education.
5. There are many constituencies for research projections. The researcher must be alert to the possible interests these various practitioners might have.
6. Mathematics education researchers can interpret the mathematical relevance and implications of their work to others.
7. The example studies presented were alert to the interests of the child and seemed to engage children in activities in which they partook willingly and from which they profited in some direct way.

These observations should be tempered by the observations of the group (some of which are in the attached material) on the researcher's responsibility in making the projects of his/her work. Some were:

a) What are the researchers' responsibilities to the users of his/her work? This entails both the "selling" of her/his ideas and alerting users to limitations or misapplications.
b) Researchers' need to make their reasons for doing the research and their assumptions in its doing clear to others.
c) Research should (a value judgment) aim to improve practice. Things one knows should be clearly and directly presented to teachers.
d) What are the ways and ethics of presenting research results so as to appropriately impact practice?

These questions indicate that deep issues remain to be explored in this area of research and the classroom. Yet the group, despite the disappointment in not at least partly resolving such issues, felt that the time was well spent in relating the 5 example studies to the issues at hand. In addition, most liked the opportunity to discuss in detail the variety of interesting research which is ongoing in Canada today.

WORKING GROUP B

COMPUTER EDUCATION FOR TEACHERS

DALE BURNETT
MARVIN WESTROM

COMPUTER EDUCATION FOR (MATHEMATICS) TEACHERS:

A Statement for the Working Group on Computers in Mathematics Education

J. Dale Burnett

The form of this report reflects a desire to encourage exploration rather than to achieve closure. Many of the points made in our discussion were as valuable for the questions they raised as for any solutions that were offered. Hence many of the "phrases" are only partially completed, simply that they may be placed on the table (or in the reader's consciousness).

The following questions and points specific to the confluence of computer technology and mathematics education were noted at various points in the CMESG proceedings (on the plane to Edmonton, at the actual working group session, or in informal get-togethers).

1. Computer technology permits the display of data structures, both arithmetically (i.e. matrices) and graphically (traditional plus newer techniques such as Tukey's Exploratory Data Analysis).
2. Glass boxes. An idea championed by Howard Peelle (University of Massachusetts) - essentially the opposite of a black box - manifested by a program or an algorithm that easily reveals the nature of the procedure.

3. Student and teacher manipulation of simulation and game models Not only by specifying parameters and observing the results but also by modifying the model itself.
4. Debriefing sessions. Particularly valuable after students have had an opportunity to run a simulation. Is likely to be of value at the conclusion of any (programming or mathematics) assignment that permits alternative approaches or answers.
5. Interactive graphics. Simple examples could include the student specifying a, b and c and having an immediate display of the corresponding quadratic graph. A number of such displays could be superimposed on one another to facilitate comparisons. A more sophisticated (technologically) example could involve the simple placing or deleting of points on a scatter-plot (using a touch sensitive surface, or even games paddles) and having the resulting correlation coefficient displayed after each alteration.
6. Video-disc technology will soon be available on computer systems.
7. To what extent does familiarity with a computer language facilitate the learning of a natural language (or a second language)?
8. What are some of the important features (from the point of view of computing science, mathematics, linguistics, psychology, education,...) of various languages such as BASIC, PASCAL, APL, LOGO, LISP...?

9. What are some of the important technological features of emerging computer systems (colour, sound, graphics tablets, touch sensitive surfaces, joysticks, music boards, high resolution graphics,...)?
10. What are some of the underlying educational philosophies of different technological approaches?
11. What are the important elements in the mathematics curriculum that could be enhanced by utilization of computer technology?
12. We appear to be improving our authoring languages (NATAL, PILOT...) for tutorial programming. We also need to improve our usage languages (APL, LOGO,...).
13. What is an appropriate role for computers in (mathematics) testing? (e.g. sophisticated drill and practice based on an analysis of previous performance, item banking, branching tests,...).
14. What features of a computer-managed-instruction (CMI) system are suitable for a school system? (See paper by M. Westrom for an explanation of CMI.)
15. What types of resource centres and clearing-houses for computer-based materials are needed?
16. One should not forget about the adjunct use of other non-computer-based materials (course manuals, worksheets, photographs, handouts,...) while using a computer. Even within a specific task, some components may be better handled in other ways.

17. There should be structure in the (mathematics) curriculum at the molar level but choice (student and teacher) at the micro level.
18. Teacher receptiveness to atypical uses of a particular program may open new avenues. For example, even a simple game such as brick-out (a computerized form of solitaire ping-pong) can lead to the exploration of various strategies and an analysis of why some strategies should work better than others.
19. Development of interesting environments for student exploration:
    - let student input any figure (using a graphics tablet, games paddle or a light pen). Have computer keep track of the relationship between area and perimeter.
    - see point #5 for other examples using interactive graphics.
    - Logo (turtle geometry)
    - APL
    - billiard ball math from Jacob's book, Mathematics: A Human Endeavour.
20. Dr. Gerald Nolting, at a previous CMESG meeting in Kingston presented a paper on the use of different proportions of orange juice and water for studying students' understanding of fractions. A natural extension of this idea to computerized screen displays, where the student alters the parameters and the shade of the resultant mixture is adjusted accordingly on the screen. Dr. Kieren indicated they are already working on this at the University of Alberta.

21. Various games are now being placed on the computer (chess, checkers, backgammon, go, tic-tac,toe, Rubiks cube,...). However the real trick is to have this set up to facilitate the writing of "strategies" for playing these games.
22. In addition to numeric manipulation, computer usage can increase the need for symbolic manipulation (e.g. algebra) and graphic manipulation (e.g. use of colour within group theory).
23. Odometer displays for different bases.
24. We need software to display program execution one step at a time (e.g. displays highlighting, perhaps by flashing or reverse video, both the line of code being executed and the change to any data structure).
25. The introduction of this technology should require mathematics educators to identify the key concepts within the curriculum.
26. Use of computer generated films to show complex mathematics phenomena (e.g. turning a sphere inside-out).
27. To date, most mathematics has been "static", in the sense that it represents only snapshots, not motion. We are now on the verge of having a genuine "dynamic" mathematics environment involving motion on a screen.
28. It should be fun. We want people to enjoy mathematics.
29. Just because computers are in the classroom, is this a prima-facae reason that they should be extensively used?

30. Some uses of computers may actually be retrogressive (e.g. some languages [Basic?] may be so discordant with actual thought as to interfere with the key conceptual processes in question - for example, matrix multiplication).
31. A distinction should be made between computing science and computers augmenting mathematics.
32. What components of the mathematics curriculum become trivialized? What areas become accessible (with the advent of computer technology)?
33. We need to develop our abilities to explore.
34. There is a perceived need for guidance on how to evaluate software.
35. Attention needs to be directed to the level of the classroom teacher. How does one use a micro in today's math class?
36. We need some exemplary demonstration programs.
37. We must be careful not to limit ourselves to today's technology.

WORKING GROUP C

ISSUES IN THE TEACHING OF CALCULUS

RALPH STALL

CMESG GROUP C 1981

REPORT

Attendance at the Group C sessions rapidly stabilized to become the following group of contributors.

Robert R. Christian
H.N. Gupta
Bikkar Lalli
Ian McDonald
Ralph Staal (Chairman)
Hugh Thurston
Yvan Roux

All participated in the discussions. The papers referred to in this report were written by the leaders of the related discussions and are included in Appendix C.

I. Agenda Possibilities

The chairman opened the proceedings with the presentation of Paper #1 (Agenda Possibilities), the contents of which had previously been available to registrants. It was not expected that all 15 suggested topics would be discussed, and they were not, but nearly all were at least touched on, sometimes as side comments to a major item.

In this report, we will treat the topics mainly in the order in which they appear in Paper #1: chronologically, there was quite a bit of back-tracking as given issues were returned to when they interacted with each other.

II. Is Calculus the same as Introductory Analysis?

It was agreed that the answer to this (admittedly somewhat rhetorical) question should be NO. Much of the material may be the same, but the focus

emphasis and motivation are different. The break-up of the (at one time) closely knit package of mathematics and science has been associated with greater compartmentalization. Mathematics has often been left standing more by itself, with a resulting shift toward Introductory Analysis which is premature, inadequately motivated, and, in spite of its intentions, not really very strong theoretically.

Calculus, in contrast with Introductory Analysis, is by its very nature heavily involved with applications - they are part of it. The value of the applications here is not "practical" (in the usual sense); rather it lies in aiding the mathematical understanding and in generating (not just applying) the mathematical ideas.

One should speak of "interactions" rather than "applications", as this would suggest more strongly a reciprocating, or symmetric, relationship between the mathematics and the areas of "application".

See also R.R. Christian's Paper #7, item 8.

R.R. Christian (see Paper #7, pp. 1,2) answered the question with a firm "No!" and suggested replacing rigor by honesty.

H.N. Gupta (see Paper #6, p.6), in the matter of rigor counselled waiting for an appropriate time, referring to the quotation:

"Give them chastity and continence, only not yet "

III. What should be done about Differentials?

This question precipitated extensive discussion. First, it had to be sharpened by distinguishing between the differential notation and its various interpretations, and by treating it in different contexts.

The chairman presented three brief papers (#2, #3, and #4). #2, titled "Logarentials" was a satirical (with apologies) account of how some of the perplexities associated with most "standard" accounts of the differential notation and concept can be transferred, by a reasonable analogy, to the subject of logarithms - the point being that the unnecessary confusion inherent in the original would (it was claimed) be more easily seen in this way.

#4, titled "By Parts", was intended to indicate by examples (the full story presumably being familiar to the Group members) that the message implied by many textbooks that integration by parts involves differentials in an essential way is entirely false, both theoretically and in terms of convenient manipulation.

#3, titled "Setting up Integrals", compared two popular ways of arriving at the integral for an area under a curve. The case was put that the method which referred to Δx's and to forming a limit of a sum was, although incomplete, a clear indication of the nature of the process involved, whereas the "differential" approach was confusing.

H. Thurston responded to the question with a substantial presentation which in expanded form, appears in Paper #5,"The Leibniz Notation". R. Staal responded in part, by claiming that the difficulties which Professor Thurston resolved by using the definition

$$dx(\alpha,\tau) = x'(\alpha)(\tau)$$

could all be resolved equally well, in his opinion, without any such definition of dx in isolation, provided one interprets only the whole of "$\int_a^b \ldots dx$," rather than the parts "$\int_a^b$" and "dx" separately.

In Paper #6, pp.3-5, H.N. Gupta viewed the use of differentials from another point of view - as tools of approximation and discovery - and urged

that they not be banished from introductory Calculus. This was not actually in conflict with Professor Staal's point of view, as the latter's comments were in a different context. The use of finite (not "infinitesimal") approximations to finite increments, followed by a limiting process was not what Professor Staal was proscribing. His concern was essentially with those uses of differentials which appeared to avoid limiting processes.

In Paper #7, item 3, R.R. Christian distinguished between the differential (manipulative) and the infinitesimal (little bit of) aspects of differentials. He emphasized the importance of honesty, heuristics and usefulness (e.g. as mnemonics).

On the whole, there was agreement that many, or most, current standard treatments of differentials at the elementary level were notably unsatisfactory but there was more to the issue than this.

IV. How should $\ell n\ x$ be introduced?

R.R. Christian, (see Paper #8 for a detailed account) pointed out the advantages of approaching logarithms by means of the function

$$\ell(a) = \lim_{h\to 0} \frac{a^h - 1}{h}.$$

This approach was not generally familiar to the Group members, but Professor Gupta commented (Paper #6, pp. 1,2,3) that it resembled the treatment given by De Morgan in 1842, and elaborated on the history of the topic, and on the desirability of making this history more widely known among our students.

Professor Christian, on subsequently referring to De Morgan's work (see Paper #7, pp.1) found some overlap with his own approach, but also some considerable differences. (See paper #9, for the relevant parts of De Morgan's paper).

V. How should one sketch Polar Graphs?

R.A. Staal raised the issue of sketching polar graphs as one which normally fits into an introductory Calculus course and which, although perhaps mathematically trivial, leads to the important question "Why do we drill students in graphing, and then ignore good opportunities to use these graphs?" He outlined a method which invoved

(a) sketching the (usually familiar) Cartesian graph of $r = f(\theta)$, or perhaps a familiar related graph on which the values of $f(\theta)$ could be easily read.

(b) plotting the Polar graph of $r = |f(\theta)|$, but dotting the portions of the curve on which $f(\theta)$ was negative.

(c) reflecting the dotted portions of the graph in (b) in the origin.

(This method has been outlined, with illustrations, in the Ontario Secondary Schools Mathematics Bulletin, V. 17 No. 3, Sept. 1981, pp. 5,6).

H. Thurston (see Paper #10) has since added a note on the use of this method in suggesting where a polar graph might involve a change in concavity.

VI. What role should historical matters play?

The main responses to this question were the following:

H.N. Gupta, in Paper #6, already referred to, exemplified the historical approach to logarithms.

Y. Roux, in Paper #11 (written jointly with M. Lavoie) presented the chairman with a paper which, although written in another context, overlapped significantly the concerns of the Group. This paper emphasized the importance of having two points of view.

(a) the mathematical: dealing with internal subtleties and logical aspects

(b) the historical: emphasizing the genesis of ideas and the influence of history on teaching.

VII Why is Calculus an Important Subject?

A member of the group asked for a response to students who asked why they spent so much time on Calculus. Why, if at all, does it have a special value?

The response was that, in addition to more obvious reasons, Calculus brings together in an interesting way nearly all the mathematics to which the student has already been exposed, and its benefits are accordingly very broad. It is also a particularly powerful tool, covering a wide range of applications.

The students should be asking 'Why do we spend so little time on Calculus?

It was pointed out that a very large portion of the time given to lecturing in Calculus is in fact devoted to matters algebraic, geometric, etc.

Mathematics teachers were urged to keep educational and mathematical matters uppermost, and to resist pressure to give courses on "Calculus for Chemists", "Calculus for Accountants" "Calculus for Economists" and others which are designed merely for training students in specialized applications. If given at all, such courses should be handled by Chemists, Accountants, Economists, etc.

WORKING GROUP D

REVITALISING MATHEMATICS IN TEACHER EDUCATION COURSES

HUGH ALLEN

Revitalalising Mathematics in Teacher Education Courses

Report of Working Group 'D'
H.A.J.Allen
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Working group 'D' met to discuss ways of overcoming the negative attitudes towards mathematics of many teacher education students. Several participants had brought along examples of topics and/or approaches that they had found to be effective in reducing students' anxiety. Through seeing some of these examples, and through the subsequent discussion, the group was able to clarify the framework within which constructive examples might apply. This report attempts to summarize the discussion that took place within the working group.

One aim of teacher education courses is to produce teachers who have (i) reasonable confidence in themselves and in subject matter, (ii) an interest in the subject matter, and (iii) the confidence to try new ideas, both mathematical and pedagogical. While some of our inservice and preservice candidates do possess these characteristics, most do not, and those that do not can be classified into two main types according to their attitude toward mathematics; namely apprehension or complacency.

Students whose attitude toward mathematics is one of apprehension (or fear, or sometimes outright rejection) have had very little mathematics in their academic background (often as little as grade 10 mathematics) and have little incentive to do any mathematics. Typically, these students are preservice or inservice elementary school teachers, and they are usually women.

The other type of student (the complacent ones) usually have a great deal of mathematics in their academic bacground-frequently including several university level courses in mathematics. Their "know-it all" attitude toward elementary mathematics makes them reluctant to try any <u>new</u> mathematics. They are products of the system, and the system rarely asked them to think. Typically, these students are secondary school teachers (practicing or intending).

To accomplish the three aims discussed earlier in this report it may be necessary to treat the apprehensive student differently from the complacent student. We want both students to do mathematics and to see mathematics as process. With the

apprehensive student it is important to reduce his/her anxiety and to have the student experience lots of success. On the other hand, with the complacent student it may be necessary to <u>increase</u> anxiety initally and to have the student experience some lack of success before he/she is willing to do any new mathematics. With both types of students we are attempting to change attitudes toward mathematics.

It is the opinion of the working group that changes in attitudes are most likely to be achieved by engaging in activities that (a) provide insight into a mathematical topic, content, or procedure that was not previously understood, (b) promote a different view of mathematics - one in which mathematics is seen as "process" rather than a set of results, and (c) provide an opportunity for the teacher to behave in such a way as to make the student (in this case the preservice or inservice teacher) more comfortable with the material. These, then are desirable characteristics in an activity and hence are to be considered in choosing the kind of activity to be used with teachers.

In addition to considering the kind of activity, it is most important to consider the content of the activity and in particular, what can be done with this in teacher education. In mathematics courses, one teaches mathematics. That is the sole content. By contrast, in teacher education one teaches both mathematics and the vehicles for teaching mathematics; i.e.,pedagogy. In addition, the teaching involves an examination of the teaching by instructor and students. This examination includes a look at the teacher's attitude toward the subject matter, toward the learner, and toward himself. In teacher education one also examines the learning that has taken place with particular reference to <u>how</u> the learning took place and to the feelings of the learners. So, in attempting to change teacher's attitudes toward mathematics, we seek activities that maximize the potential to change attitudes (as described in the previous paragraph), and that maximize the kinds of things we can and should do in teacher education (as described in this paragraph).

The participants in this working group tend to favor the "investigation" activity and particularly those investigations that are reasonably open, since these activities seem to provide the maximum opportunity to (i) view mathematics in a different light, (ii) see the possibility of the teachers behaving in a less authoritarian manner and more as a learner and a resource person, and (iii) examine one's attitude toward the teaching that has taken place.

Several investigations and other activities were discussed by the working group. Some of these activities appear 'non-mathematical' (i.e., devoid of number) and are designed to demonstrate some of the processes of mathematics in a non-threatening form. Examples are "word chains" (described in Appendix A of the Report of Working Group 'A', CMESG conference Proceedings, Kingston, 1978), and the example "Animals" in Appendix D of this report. Other examples considered by this group are also included in Appendix D of this report.

SPECIAL GROUP P

THE PLACE OF GEOMETRY IN THE ELEMENTARY SCHOOL

DIETER LUNKENBEIN

Canadian Mathematics Education Study Group (CMESG)
Groupe Canadien d'Etude en Didactique de Mathematique (GCEDM)
1981 Meeting

University of Alberta, Edmonton
June 5-9, 1981

Special group P: The place of geometry in the elementary school
(D. Lunkenbein)

"Geometry at the present time does not seem to be an important part of mathematics teaching. Is plays a minor role in official curricula and an even smaller role in most classrooms. This situation seems to be in sharp contrast to the intrinsic worth attributed to geometry teaching, on the one hand, and to the potential of informal geometry as a valuable vehicle for arousing the interest and strengthening the confidence of both teachers and children, an the other hand".[1] Many reasons may be given for such a situation. Certainly, the clear distinction between arithmetical and geometrical activities made in mathematics teaching from the very beginning on is one of the most important reasons for such a misconception of geometry in school curricula. Arithmetical and geometrical activities are but two different ways of approaching mathematical phenomena or problems and, in most cases, both types of activities are inseparable. In the course of acquiring elementary mathematical knowledge, arithmetical representations of geometry and geometrical representations of arithmetic go hand in hand and, from this point of view, the separate programming of geometrical and arithmetical activities appears the be rather artificial and prejudicial for the learning of mathematics.

1. Geometry in the Elementary and Junior High School Curriculum, Report of Working group C, Proceedings of the 1980 Meeting of the CMESG at Laval University, CMESG, Montréal, March 1981, page 74.

For a balanced mathematical education at the primary level, it seems to be of great importance, that such historical, epistemological and conceptual links between geometry and arithmetic be made explicit particularly to the teacher, so that they may be taken into account in classroom activities.

It was the aim of this group to investigate the possibility to communicate such links directly to the teacher in order to motivate him/her to integrate geometrical activities in day-to-day classroom work. Based an the outcomes of last year's meeting (Report of Working Group C, Proceedings of the 1980 - meeting), the original production or the new presentation of teaching material was to be considered in view of such genetic links between geometry and arithmetic with the intention to produce an appropriate document which would be accessible for and attractive to elementary or junior high school teachers.

Initially, the discussion centered around an exploratory activity concerning polygonal shapes composed of congruent equilateral triangles[1]. The example was intended to show a possibility of the gradual unfolding of a conceptual context by both geometrical and arithmetical means through continuous investigations over several years of schooling. Among the topics discussed were the nature and originality of possible themes, types of activities that would be most suitable for the purpose, general goals of such activities and possible formats of presentation. It was generally felt that there is no shortage of examplary subject matters or themes for such an entreprise, but that the most difficult task lies in the way of communicating the important ideas to the teacher. Such communication would be the most effective if presented in a ready-to-teach form and well commented as to the investigative

1. D. Lunkenbein, Deltagons: Shapes and Numbers, Département de mathématiques et d'informatique, Université de Sherbrooke, Sherbrooke, Québec, mai 1981.

character of the activity, the nature of the links between geometry and arithmetic, the possible insertions of the activity into current curricula and the general goals to be attained. The format of presentation may vary a great deal; the teacher's guides of the South Nottinghamshire Project[1] were mentioned as possible examples.

These discussions did seem to stimulate the enthusiasm and the creativity of the participants to a point that we agreed to give it a try and to work on it during the coming year. For this purpose, first-draft papers would be sent to the group coordinator by the end of July 1981 who would then in turn circulate these papers among participating authors in order to stimulate the discussion. Hopefully, we will be able to finalize these papers at the next CMESG - meeting and to publish them for wider circulation.

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1. Bell, A., Wigley, A., Rook, D., Journey into Maths, Teacher's Guides 1 and 2, (The South Nottinghamshire Project). Bishopbriggs, Glasgow: Blackie and Son, 1978, 1979.

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PANEL GROUP R

AN EXAMPLE OF MATHEMATISATION: THE RUBIK CUBE

FERNAND LeMAY

CANADIAN MATHEMATICS EDUCATION STUDY GROUP

UNIVERSITY OF ALBERTA, EDMONTON

(June, 5-9, 1981)

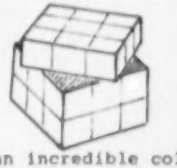
AN EXAMPLE OF MATHEMATISATION: THE RUBIK CUBE

F.Lemay

(Outline of Edmonton's Talk)

THE CUBE.-

Rubik's cube appears to be a clever setting of *27* smaller cubes or *"elements"* (or is it *26*, since the interior one is never to be seen?) whose visible faces borrow their colors to the *6* faces of the main cube (in its natural homogeneous form) and whose *9* "slices" can be rotated globaly and independantly



thus giving rise to an incredible collection of *multicolor forms*.

THE PROBLEM.-

The problem? Of course to find its way back out this labyrinth from any multicolor form or, conversely, to reach any pattern that might be proposed.

Or, if some multicolor forms are out of reach, to invent *accessibility criteria*. To determine the *orbit* of any form, that is the set of all its accessible forms. To describe ways of connecting any two forms.

To find didactical applications. To gather eventually algebraic subproducts.

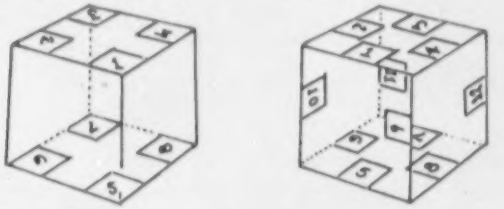
To proceed to an epistomological prospection.

Etc.

SOME NOTATIONS.-

The structure of the cube does not allow the small central squares of any of the *6* faces of the main cube to leave their positions; therefore their constellation of colors acts as a *reference* and *coordinates system* for all elements of the cube: those lying at the vertex (*vertex elements*) shall be determined by *3* colors, those lying at the middle of edges (*middle elements*) by two, and of course the central elements of various faces by one only.

For a given element, the position is determined by that of any of its particular faces. Twenty-four positions can thus be assigned to a vertex element and twenty-four others to a middle element. Let's index these two sets of positions arbitrarily by giving ourselves a *"monitor"*



All mobile elements of the cube then also acquire an index: that associated with their natural position. The set of vertex elements can then be written (†)

$$S_0 = \{1,2,3,4,5,6,7,8\}$$

and the set of middle elements as

$$A_0 = \{1,2,3,4,5,6,7,8,9,10,11,12\}$$

Other vertex positions derive from these by rotations of one third of a turn ("screwing in" the cube) thus leading to the set of *"polarised positions"*

$$S = \{1,1',1'', \ldots 8,8',8''\}$$

Similarly the set of all possible positions of middle elements shall be denoted

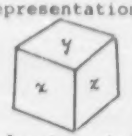
$$A = \{1,1',2,2', \ldots 12,12'\}$$

(†) *S and A being the initial letters of the corresponding french words "sommet" and "arête".*

INVENTORY OF ACTIONS.-

In order to generate all multicolor forms, only the 9 rotations of the 3 slices are available; but these reduce essentially to the 6 rotations of the external faces since the rotation of a central slice is equivalent to rotations (in opposite directions) of the two parallel slices. The rotations of these external faces (quarter of a turn, screwing in the cube) shall be named after the characteristic colors of the external faces.

In every representations that follow



x, y, z shall always denote rotations of the left, of the upper and of the right faces respectively, while x', y', z' shall refer to the opposite faces. Finally the inverse of a rotation x, shall be denoted simply $\bar{x}$.

TRANSITIVITY IN S AND A.-

Our six rotations allow any vertex element to move freely about the cube, but once an element has reached its target, the three colors liable to push it further have to be *"confiscated"*. Taking this strong constraint under consideration, it will not be possible, in general, to assure more than the *"3-transitivity"* (that is the transitivity of triples of vertex elements in the set of all polarized positions) which, as experience proves, *is* far from our objectives.

Perhaps should we have started by carrying the middle elements first? This time we would reach 3-transitivity in A_8 but not in the polarized set A.

COMPLEX ACTIONS.-

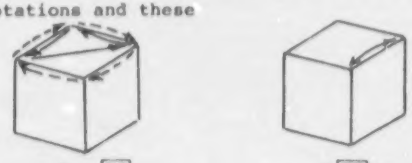
We might conjecture that all accessible multicolor forms must remain "within 3-transitivity", but once again experiments shall very soon destroy that conjecture.

Solutions seem out of reach!

While nothing else than our six rotations is available, we must become aware of the fact that a "chain of rotations" can be seen as a new *complex action* having eventually new properties. Thus the composites of two "adjacent" rotations, or of the three rotations "around a vertex", or of the four rotations "around the walls"

xz $\qquad$ xyz $\qquad$ $\boxed{xz} = xz'x'z$

give rise to new cycles, and the actions resulting from "conflicts" between rotations and these



y, $\boxed{xz}$ $\qquad$ $y\cdot\boxed{xz}$

finally provide us with the means to realize complete transitivity in S_8 (that is 8-transitivity on the set of vertex elements, neglecting polarization).

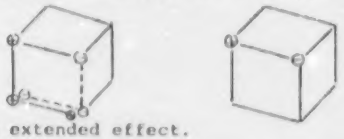
GYRATORS. DIPOLES.-

A more detailed description of $a=xyz$, for instance, reveals that, while 4 elements circulate through 4 stations, 3 others "spin" on the spot; consequently by iterating properly that action, we can retain the *"gyrator"*

$a^4 : S \longrightarrow S$

(• indicates a spin of one third of a turn).

Now combining this gyrator with its inverse, one gets a *"dipole"*



which has a less extended effect.

It is now possible to control the polarization of all but one vertex and to stand with the firm conjecture that *it is impossible to rotate a single vertex without affecting others.*

CATASTROPHE.-

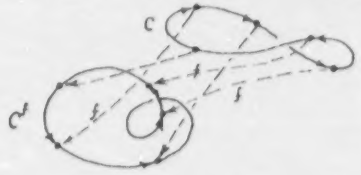
We could expect the structure of the cube to be so strong that the complete relocation of all vertex elements would necessarily carry the intermediate or middle elements back to their natural positions; but experience denies such a conjecture and we are confronted with *the problem of acting on these intermediate elements while leaving the vertex absolutely inert.*

THE DISCOVERY OF MALLEABILITY.-

Having gradually modified the underlying substance of our investigations from cubic elements to rotations, to complex actions, - we still must go one step further toward a new vision and elect a new underlying *"matter"* on which to act.

The *cyclic paths* followed by the various elements of the cube are also "objects", complex objects, which can be submitted to our actions and which will thus reveal their *"malleability"*.

If a cycle C for instance undergoes a transformation f



then there will result a new cycle

$$C^f = \bar{f}Cf \quad \text{(read from left).}$$

In particular the rotation z, transformed twice by the action xz, becomes the rotation x, so that a "conflict" between them

$$\bar{x} \cdot (z^{xz})^{xz}$$

shall neutralise their mutual effects on S while it will continue to act on z

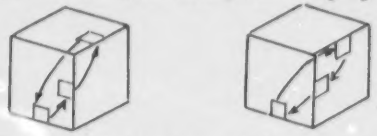
giving rise to a *circular permutation of three middle elements.*

A "UNIVERSAL" MAP.-

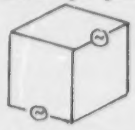
Taking malleability into account this last cycle enables us now to move freely on the *"universal map"* connecting all middle elements

and to execute any *even* permutations of A_8 that we might wish to realise.

As for the polarization of middle elements, malleability also enables us to transform the preceeding cycle in a similar one



from which a "conflict" develops giving rise to an *"alternator"*.



We therefore have gained control over *all permutations*

in S_8, over all even permutations in A_8 and over the polarisation of all elements but one vertex and one middle elements.

From this we can deduce the existence of at least

$$8\times 12!\times 8!\times 2^{11}\times 3^7 = 43.\ldots\times 10^{18}$$

accessible multicolor forms.

CONFIRMATION OF THE CONJECTURES.-

Any multicolor form exhibits a particular transformation of the original cube and they therefore determine a certain subgroup

$$\mathcal{C} \subset \mathfrak{S}(A\times S)$$

of the group of permutations of $A\times S$ which I shall call the *"group of polarised permutations of the cube"* (†).

Among these "targets", only those are *accessible* which belong to the subgroup generated by our six rotations

$$\mathcal{A} = (x,y,z,x',y',z')$$

The action of this group can be "transferred" to any target; which amounts to say that it is *"transportable"* or, technically, that *$\mathcal{A}$ is a normal subgroup of $\mathcal{C}$*. The quotient group $\mathcal{C}/\mathcal{A}$ then represents the *class of orbits* of the various multicolor forms. In other words, accessibility or reducibility from one form f to another g, is expressed algebraically by

$$f \equiv g \pmod{\mathcal{A}}$$

Our proposition is the following:

Theorem.- $\mathcal{C}/\mathcal{A} \simeq Z_2\times Z_2\times Z_3$

In order to prove this, let us construct an homomorphism

$$\Gamma : \mathcal{C} \longrightarrow Z_2\times Z_2\times Z_3$$

(which we shall call the *characteristic of multicolor forms*) defined in the following way (with respect to an arbitrary monitor).

(†) *Consisting essentially in a pair of permutations of A_8 and of S_8 coupled respectively with two maps (the "alternance" and the "gyration")*

$$\begin{cases} \alpha : A_8 \longrightarrow \{0,1\} \\ \gamma : S_8 \longrightarrow \{0,1,2\} \end{cases}$$

expressing the eventual reorientation of the smaller cubes.

Definition.- For any multicolor form $f\in\mathcal{C}$, π shall be the parity of the multicolor form, that is the parity of the permutation induced by f on $A_8\cup(S_8+12)$ (†) Secondly Θ shall be the sum mod 2 of the values of the function α. Finally Σ shall be the sum mod 3 of the values of the function γ. The characteristic is then defined by $\Gamma=(\pi,\Theta,\Sigma)$.

One shows first that Γ is independant of the particular monitor that we have chosen and that it is an homomorphism

$$\Gamma(fg) = \Gamma(f) + \Gamma(g).$$

The rest of the proof amounts to showing that the kernel of Γ is $\mathcal{A}$.

SOME COROLLARIES.-

The computation of Γ is very simple and permits us to formulate a certain number of corollaries:

1) There exists an *"auto-dual quantum"*, ultimate germ of all actions (a double transposition acting on a pair of vertex elements on one hand, and on a pair of middle elements on the other hand).

2) The simplest of *"pure cycles"* are permutations on three elements of one kind.

3) The shortest gyrator is a dipole.

4) The shortest alternator acts on a pair of middle elements.

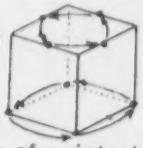
5) There exists a *"universal cycle"* involving all elements (that is a double cyclic permutation of the *8* vertex and of the *12* middle elements respectively).

6) For any sequences of distinct vertex elements and of distinct middle elements, there exists a double cyclic permutation of these sequences at the sole condition that the total number of elements in these sequences be even.

(†) *The use of S_8+12 amounts to reindexing temporarily the set S_8 in order to eliminate overlappings between A_8 and S_8.*

FISSION.-

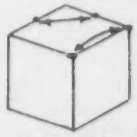
As an illustration, let us observe the striking "explosion" of the rotation y under the action of $\overset{\bullet\bullet}{z}y'\overset{\bullet}{z}'$ (where $\overset{\bullet}{z}$ refers to the rotation of the *central slice* which parallels z, ...)



leading to the *fission of y into two disjunct rotations in A and S.*

THE AUTO-DUAL QUANTUM.-

The occurrence of this first fission into *"dual actions"* is the starting point for the search of other actions acting *"symmetrically" in both A and S*, and leading to the discovery of a *"quantum" of action*



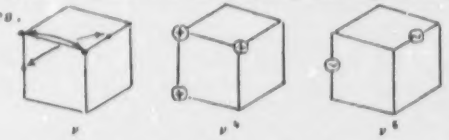
whose orbit generates all accessible multicolor forms.

ULTIMATE GERM OF ACTION.-

Finally by observing the fission of the rotation y more carefully, one is lead to the discovery of an other quantum acting in $K_8=A_8 \cup S_8$:

$$\nu = (y \cdot y^{zx'})^{z'} \cdot \bar{y}$$

The orbit of ν has the most interesting property of producing *all accessible permutations of A_8 and S_8, all gyrators and all alternators.*



ν $\qquad$ ν^4 $\qquad$ ν^6

We have thus obtained a basis for a *"dual treatment"* of the mathematics associated with the Rubik's cube.

Laval University, Quebec.
June 1981.

PANEL GROUP X

MATHEMATICS AND LANGUAGE

MARTIN HOFFMAN

Panel Group X: Mathematics and Language

Panel members: A. J. Dawson
W. C. Higginson
Moderator: A. E. Hoffman

The stated purpose of the Panel Groups was "to present themes which, if there is enough interest, may become the subject of working Groups at subsequent conferences". Panel Group X, attended by approximately 25 CMESG members, consisted of presentations by the panel members, followed by questions and comments by those in attendance. This report will briefly highlight the main themes of each presentation and the following discussion. Papers prepared by the panel members appear in the appendix of these proceedings, and should be consulted by those interested in pursuing these topics in greater depth.

Bill Higginson's presentation, entitled "Arguments for the Consideration of Language by Researchers in Mathematics", after offering several interpretations of "X", offered an overview of several areas for further study: The relationship of mathematics and language with respect to context, communication, content and cognition. Each aspect was briefly discussed and supported with examples and relevant bibliography.

Sandy Dawson's presentation, entitled "Words triggered by Images, Images triggered by Words", focused on one aspect (imagery) of one of the areas (cognition) of possible study noted by Bill Higginson. The crucial role of images in the learning of mathematics was discussed. It was argued that

teachers must work with the images possessed by their students before the words of the teachers can be used to reliably transmit images to them.

At this point the entire group was engaged in an image making activity led by Dawson. They were asked to imagine a lemon and then to perform various manipulations on it. The discussion which followed indicated the wide range of images generated by the single set of instructions.

Several well-considered points were made during the discussion following the two presentations. Among those concerned with the question of extending the Panel Group to a future Working Group were the following:

1. For the purposes of CMESG Working Groups, visual imagery and language (in the general sense of Higginson's presentation) should be separated due to time considerations.
2. The general considerations presented by Higginson would need more focus to fit the restrictions of the Working Group format.
3. The potential for group participation (as in Dawson's presentation) was favorably noted.
4. The need to have working mathematicians attend these (projected) Working Groups was expressed.
5. It was hoped that one of the future invited speakers would focus on these topics.
6. Several participants suggested that, if such a Working Group is established, sufficient lead time be given by the Working Group leaders to allow prospective participants to read relevant articles and to gather examples from their own teaching experiences for presentation in the group.

PANEL GROUP Y

LES RELATIONS ENTRE L'HISTOIRE ET
LA DIDACTIQUE DES MATHEMATIQUES

BERNARD HODGSON

Groupe de discussion Y.

LES RELATIONS ENTRE L'HISTOIRE ET LA DIDACTIQUE DES MATHEMATIQUES.

Bernard R. Hodgson

Certains aspects des relations existant entre l'histoire des mathématiques et la didactique des mathématiques ont été présentés par les trois invités. Leurs textes figurent en appendice. Louis Charbonneau (Université du Québec à Montréal) s'y questionne sur les leçons que l'étude du développement historique des mathématiques nous fournit à propos des stratégies d'apprentissage avec les enfants; un parallélisme y est fait, par exemple, entre certaines conclusions tirées d'une expérimentation avec les enfants à propos de la numération et une étude historique de ce même sujet. Hara Gauri Gupta (University of Regina) propose qu'à l'instar des étudiants en philosophie, histoire ou littérature, les étudiants en mathématiques devraient, eux aussi, avoir l'occasion de remonter aux sources et de "lire les classiques". David Wheeler (Concordia University) présente une série de commentaires sur certains aspects généraux du lien histoire-didactique. Il conclut en soulignant que non seulement une démarche didactique peut tirer profit d'une vision historique, mais également, de façon inverse, le travail de l'historien peut être enrichi par les interrogations présentées par le didacticien.

Les trois présentations ont été suivies d'un échange de vues avec les participants. On y a fait ressortir divers avantages d'une connaissance de l'histoire pour l'enseignant. Par exemple, l'histoire des mathématiques permet de voir celles-ci comme une science en évolution et non pas comme une science achevée. Une telle perception aide l'enseignant à voir les mathématiques de façon dynamique et le rend sensible aux besoins pédagogiques des étudiants quant à l'importance de créer les mathématiques pour eux-mêmes. Cet acte de création est souvent l'objet de nombreux échecs partiels; mais ces difficultés de parcours trouvent leur pendant dans la démarche même des grands mathématiciens d'autrefois et fait partie intégrante du processus de découverte des mathématiques. Les enseignants de tout niveau, même du primaire, peuvent "humaniser" l'enseignement des mathématiques en introduisant judicieusement certaines parenthèses historiques ou, à tout le moins, certains éléments faisant partie du bagage culturel et "folklorique" de notre civilisation: division de l'heure en 60 minutes, de l'année en 12 mois, etc.

Plus d'une douzaine de participants ont pris part au groupe de discussion.

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APPENDIX A

CONTRIBUTIONS FROM WORKING GROUP A

WORKING GROUP A

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Walter Szetela

Charles Verhille

David Wheeler

Working Group on Research

A. Work I am doing:

I am currently engaged in an extended review of the research literature related to mathematical abilities, with a view toward the construction of better theoretical perspectives on such abilities and their development. I anticipate that my upcoming term as JRME editor will delay my research efforts for a few years, but I plan to return to the topic when my term ends.

B. Reactions to the working group's activities:

It is always interesting to hear firsthand, non-technical accounts of recent work, and I found the presentations in our group especially interesting. I am pleased to see so many attempts to involve teachers actively in the research process. I hope that the reports we heard signed a movement- at least in Canada- away from the idea that research should be disseminated to teachers and toward the idea that they should be partners.

I was sorry that we didn't have more time to discuss what one might call the "theoretical underpinnings" of the research including why the research was undertaken and what the research saw as the relevance of the research to theory construction.

I am interested that no one in the group is apparently pursuing research on teaching. I would have liked to hear about that if someone had ideas on it.

Finally, I am interested in the phenomonon of "cultural lag" in research as evidenced by the contortions that had to be undergone for finding agencies (who have an outmoded concept of research) in order to do the research considered important. It's a serious problem.

Jeremy Kilpatrick

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I.

A. Current Research Work:

Investigation in a mathematical context of the sequence in which logical operations (Piaget's 16 binary) are learned-examining learning hierarchies (nonlinear). Of serious concern are the differences in interpretations given by kids of linquistically different and logically isomophic statements. What are the effects of mathematical content and context? What are the effects of language? How important are these logical operations and the verbal connectives in mathematical thinking?in the mathematics classroom?in reading mathematical material?

B. Working Group Report:

Our early discussion suggested that our task was to examine the communication of research results. Several researches were viewed in light of their communication techniques. Claude Gaulin's project indicated the many audiences involved.

We were not very clear on what gets communicated to whom and furthermore Wheeler suggested that research is not very often persuasive, or that it does not have the power to be persuasive? In other words, should we go out and "sell" research results, or "pack it in?" In fact, I suspect, neither of these is the entire answer, although some version of the former must be adopted if we have anything at all to offer.

The difficult communication problem is to convince teachers that research has something to offer. The various examples at the conference of researchers interacting with classroom teachers suggests a direction.

The presentation by Kilpatrick came closer than anything we did in the Working Group to provide closure for the topic, although the initial questions remain unanswered. Nevertheless, it was a worthwhile experience and provided some new "glasses."

Lars Jansson

II.

A. Research and the Classroom:

In reporting to the group on the current (uncompleted) project, "An investigation of the mental operations of high school students in solving mathematical problems", I made the following points:

1. We experienced difficulty in identifying "mental operations" and found it easier, though less satisfactory, to talk of "strategies".

2. How should one treat the evidence provided by "talking aloud"? Is it reliable, as Simon suggests, or should it be treated with considerable caution as some other researchers state?

3. It seems fruitful to regard problem solving protocols as composed of "episodes" rather than attempt to interpret the significance of each spoken contribution.

4. In analysing protocols we found that some subjects used language in characteristic ways, though there was no obvious correlation between the level of the language used and the sophistication of the solution.

5. We have experimented with different roles for the interviewer, but we tend now to feel that the benefits of the "clinical interview" technique are best realized when the interviewer takes a positive role in the situation.

The question that remains is, "What contribution does this research make to mathematics teaching in the classroom?" The handfull of teachers involved in the project have frequently been surprised by their students' responses, and this has certainly opened their eyes to some things they were not noticing before, but what good can emerge for others who did not take part? There is no simple and direct answer, and perhaps for a research of this type it would be naive to expect the consequences to be clear and immediate. It is possible that the research has more to say to the community of researchers, particularly in bearing witness to the complexity of the problem solving process even in such relatively controlled situations as were examined.

B. Comment on the work of the group:

In commenting on the work of the group I must first say that I found the discussions stimulating and provocative of a number of new insights, not all of which I can formulate with clarity yet.

At first I felt that the "presentations" by several members did not address the main question (nor did they, in general), but as time went on I perceived that it was possible to take the content offered by the speakers and use it as a "medium" for thinking about the question of applicability oneself. Indeed at times it seemed as if discussion was proceeding on three levels simultaneously:

a) a description level on which the content, style and methods of particular projects was discussed

b) an analytic level on which various implications were teased out (and an interesting point here was that it occasionally seemed as if the "onlookers" saw things that the researcher had not)

c) a "meta" level on which the researches were related to a wider framework of educational intentions.

By itemizing the following I'm giving a false impression, but the list contains some of the "conclusions" I drew from the group's work:

1. It is tempting to identify "research" with "funded research project", but the real research questions in education are extremely challenging, may take years to study, and will never find total resolution. A "research project" is unlikely to produce anything important unless it is powered by one of these bigger questions.

2. Some "research projects" are quasi-political events, designed to persuade someone to an opinion or an action rather than to uncover the facts of the case. They may be tactically necessary (since the administration of education is a political business)but they should not be confused with research.

3. The central questions of mathematics education research almost always appear to be epistemological - <u>how</u> do students successively master counting, <u>how</u> are the aspects of rational number related to each other in experience, <u>how</u> does the structure of a problem suggest or inhibit adoption of a strategy, and so on.

4. One of the (unintended) results of research is to give the researcher an enhanced view of children's (people's) abilities - they always know more and are far more inventive than any curriculum (and almost any teacher) presupposes. Indeed I tend to think that <u>this</u> is the core of the case for giving all teachers some experience of research activity.

Of course enquiry is important, vitally important, but one has first to be convinced that there are questions to enquire into. The layers and layers of assumptions, prejudgments and prejudices that are embedded in and passed on by the educational structure have first to be pierced before anyone will start asking any significant questions.

5. A question I am left with and which nags me because I cannot get close to a resolution is the extent of the responsibility of the educational researcher for what other people do with his work. The much clearer case of scientific research has, as is well known, generated much discussion of the nature of scientific responsibility, yet I don't hear the question discussed by educational researchers (and it only came up tangentially in the group's discussions). Yet perhaps the misuse of inconclusive educational research (since much of it is inconclusive, in fact) may be no less serious than the misuse of scientific "discoveries". Is Thorndike, say, entirely blameless for the reductive uses to which the educational system has put his theories and results? Hasn't Piaget offered teachers a range of wonderful excuses for failing to teach their students?

David Wheeler

June 12, 1981.

III.

A. Research Activity:

Grade level: 7

Duration: September 1981 - June 1982

Numbers: 30 teachers, 750 students

Main Purposes:

a) teach problem-solving strategies (PSS) to teachers for implementation in class

b) familiarize teachers with calculators

c) compare performance of students in PSS groups with those in PSS-Calc. groups

Procedure:

Randomly assign classes to PSS or to PSS- Calc. groups. Provide instruction on PSS through in-service. Assist teachers in implementing PSS and calculator use in their classes. Give Pre-test, Mid-tests, and Post-tests.

Tests:

a) Pre-test: Standardized test of B.C. Ed. Ministry
b) Mid-test 1: P.S. test with paper and pencil only
c) Mid-test 11: P.S. test with calculators
d) Post-test 1: P.S. test with paper and pencil only
e) Post-test 11: P.S. test with calculators

Expected Outcomes:

1. Better P.S. activities in participating classes

2. No significant difference on paper and pencil test

3. On story problems, PSS-Calc group / P.S.S.

4. On strategy problems, no significant differences, when calculators are used.

B. Research Group:

The sessions on Research and the Classroom were relatively unproductive with respect to answering questions on putting research results into the classroom. This is no fault of the group. It is simply a very difficult problem. The reports of some research activities were quite interesting. It was useful to learn about the kinds of research being done in Canada. More important was the opportunity to interact with other Canadians in a group small enough to present a wide variety of experiences and ideas.

Walter Szetela

IV.

A. Mes travaux:

a) Recherches sur la numeration et la notion de fonction en prenant pour principe directeur qu'il existe un parallele entre l'évolution d'une notion ou d'un concept et les étapes de l'acquisition de celle-ci.

b) Experimentation dans une école élémentaire, d'un projet de perfectionnement basé sur la resolution de problème. Ce perfectionnement vise à une intégration des mathématiques et des sciences. Il a pour caractéristique d'avoir un contenu et un mode de fonctionnement déterminés par les enseignants.

B. Apports des travaux du groupe:

a) Connaissance des résultats et de la methodolgie des recherches de certains collèques (Kieran, Harrison, Wheeler, Lunkenbein, Gaulin).

b) Les discussions m'ont permis de conceroir quelques activités destinées à motiver les futurs enseignants à l'élémentaire à acquerier une connaissance mathématique, en leur montrant comment une telle connaissance permet l'analyse des comportement de l'enfant en situation d'apprentissage (spec. Kieran, Lunkenbein).

c) Quelques reflexions sur les façons d'induire les enseignants en exercice à devenir eux-même un peu des chercheurs.

d) Acquisition d'une meilleure vision de ce qui, dans une recherche, peut atteindre la classe, et ce qui n'est pas destiné à l'atteindre.

Louis Charbonneau, U.Q.A.M.

V.

APPENDIX B

CONTRIBUTIONS FROM WORKING GROUP B

INSTRUCTIONAL USES OF MICROCOMPUTERS

Marv Westrom
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With the recent proliferation of microcomputers in schools, the term "CAI" is heard frequently. This term has been in use for about fifteen years, and over this time the range of things that it can mean has changed and grown. At the same time, many equivalent terms such as CAL (Computer Assisted Learning). CBT (Computer Based Training), and CBI (Computer Based Instruction) have come into use.

In 1971, A.B. Salisbury tried to sort out the many meanings of CAI and to restore a more precise and more useful interpretation of this term. (He would have used it to stand for Computer-Administered Instruction, rather than the usual Computer Assisted Instruction.) However he was not entirely successful; "CAI" and related terms currently have no generally accepted single meaning.

It is not necessary that the terms we use to describe what we do in using computers for instruction be absolutely precise. However, it is useful to understand the various classes of things that can be done; which are most useful, and why. In identifying the things that can be done, it is natural and useful to put labels on the various classes.

There are many uses of computers in schools: administrative uses, research uses, uses in libraries, guidance and counselling, testing, management, et cetera. Some aspects of these uses are feasible only on large computers, and some uses have become possible only because of the availability of microcomputers. The purpose of this paper is to examine (in detail) only one of these - the INSTRUCTIONAL use of MICROCOMPUTERS in the school. When only large computers were available, there were only two main classes of instructional use of computers: CAI and CMI (Computer Managed Instruction). These uses are viable with microcomputers, but several new uses are possible: Presentation Aids, Prototypes, Data Management, and Utility Programs. Each of these six subclasses is considered below.

1) COMPUTER ASSISTED INSTRUCTION

Computer Assisted Instruction (CAI) is used here as a generic term to include all uses of microcomputers where a student sits at a terminal and goes through a previously designed instructional interaction with the computer. It is meant to include instructional interactions labelled as CAL, CBT, CBI, and some kinds of CMI (where all computer work is done while the student is at the terminal). There are commonly three to five "kinds" of CAI that can be distinguished (see reference 2, for example). The three most common kinds are Drill and Practice, Tutorial, and Simulation CAI. These, along with two other kinds are discussed below.

Drill & Practice CAI

CAI Drill & Practice (D&P) programs are used to help the student Drill or Practice some particular concept or skill that has already been learned (often by other means).

The most obvious example of a D&P program is one which students might use to practice their multiplication tables. The computer would present a question such as

"How much is 6 x 7?"

The student would then try to answer this within a given time limit, perhaps being given two chances. If he was successful, the program would just go on to another question of the same form:

"How much is 8 x 3?" ... and so on.

The program might "remember" which facts the student had difficulty with and give him extra practice on these. It might also keep a "score" - how many questions tried, how many correct, the time taken, etc., for the student and/or his teacher. It might use this score to move the student "up" or "down" through various levels of difficulty. Other examples of D&P programs are: a typing drill program, French vocabulary drill, a drill on the capital cities of the countries in South America, a drill on important dates in 18th century history; the Vancouver School Board has a drill on questions from the written portion of the Provincial Driver's License Exam.

There are obvious advantages to D&P CAI programs. For one thing, they are relatively easy to produce. In a few hours, one can program a drill routine which a large number of students can use for hours of practice. Such programs are excellent for helping students to learn things which are retained by rote memory.

The disadvantages of D&P programs are: a good portion of the things that students are expected to learn in school are not to be learned by rote; rather they are to be understood. D&P programs would not (for example) help students learn how to write a paragraph properly or understand the relationships among the causes of the Second World War.

Tutorial CAI

The Tutorial CAI program, (as the name implies) is intended to cause the computer to behave like an intelligent TUTOR. The computer participates in a two-way interaction with the student, through which the student (hopefully) learns something.

When people think of CAI, they most often think of Tutorial CAI programs. Usually, the computer presents one or two screens-full of information for the student to consider (or learn). Then a question is posed. If the student gets the question right, he goes on to the next section of the program (and the next item to be learned). If he gets it incorrect the program provides some "remediation" information - reminding the student of some necessary background information or perhaps clarifying the question. Better Tutorial programs are very thorough in their evaluation of student responses. They discriminate between several classes of

correct and incorrect answers and provide help and direction most appropriate for each individual student (depending upon how he answers the questions).

Good Tutorial CAI programs have been able to demonstrate an improvement in the rate at which students learn. Improvements of 20 to 30% over traditional instruction are not uncommon. Using Tutorial CAI enables instruction to be more closely tailored to individual student differences. In particular:

RATE: Students can proceed as quickly or as slowly as they wish (this could be a disadvantage).

CONTENT: A student sitting down to a computer has a choice of content from among all Tutorial programs available. Better tutorial programs allow the student to investigate subtopics about which they have a particular interest; so that even taking the same course, students might receive different information.

DIFFICULTY: Good tutorial programs can adapt the difficulty of the instruction and the questions posed to the students' demonstrated ability. Poor students feel successful because they can answer the questions; good students are challenged by the difficulty of the material.

SCHEDULE: If instruction is available in Tutorial CAI form, students can take that material whenever they wish. They are not restricted to learning when the teacher and the other students are available.

Unfortunately, there are some disadvantages to Tutorial CAI programs. For one thing, they are rather difficult to do well. It is quite easy to develop Tutorial programs which merely turn the pages of a text for students. This is a waste of authors', teachers' and students' time as well as a gross misuse of computers. Having text in its common printed form is far superior to having it doled out under program control.

While it is nearly trivial to constantly ask questions of students which can be answered by "YES", "NO", "A", "B", "C", etc., this does nothing for their ability to use English; it reinforces their propensity to not pay attention to the explanations or the questions (they merely "guess" until the computer tells them the answer); and it is boring. Properly analysing student responses is very difficult. It is usually accomplished with sophisticated response-analysis functions provided by high-level authoring languages. Unfortunately, these systems are not available on microcomputers (yet). The versions of PILOT I have seen do not provide sufficient power to do acceptable response analysis (although some attempt can be made through BASIC or PASCAL).

Students taking tutorial CAI programs typically work at very different speeds. Early literature reported a spread of rate by ability in the order of 4 to 10 times. (That is, one student might complete a Tutorial in one hour where another would take ten hours.) This can cause severe scheduling problems in a school where students are expected to complete tasks in approximately the same time.

A final disadvantage of Tutorial CAI programs is that they tend to be very large. Because of the many problems that a student might encounter when trying to learn a concept, many remedial sections must be included. If two or more levels of remediation are provided, the size of the program goes up geometrically. This is the main reason why common statistics predict that it takes about 200 hours of design/author time to produce 1 hour of working CAI (Tutorial) material.

CAI Tutorial programs are most effective when used for regular remediation and enrichment purposes. They can be used to help slow students catch up to the rest of the class (at the investment of extra time on their part), and keep faster students challenged and usefully engaged. For main-stream instruction, the investment in Tutorial CAI is justified only when very high-quality instruction is needed, when there is a requirement that the instruction be precisely repeated, or when the instruction is to be taken by a large number of students (200 or more). There are many circumstances (particularly when students are being paid a salary while they are learning) in which Tutorial CAI can be more cost-effective than traditional instruction.

Simulation CAI

Simulation CAI programs attempt to provide a computerized model of a real or artificial system. Through using such a CAI program, the student should obtain an experience analogous to being in a corresponding "real world" situation. The fact that the experience is "analogous" has the obvious disadvantage of being vicarious; however, it can have a number of advantages as well:

TIME COMPRESSION: If the real world situation is one which normally takes place over relatively long periods of time, the simulation can "compress" this time factor so that the student can more easily observe changes. For example, a simulation of the factors involved in the ecology of a lake.

TIME EXPANSION: Similarly, if the real world action normally occurs in very small time intervals, the simulation can slow the process down so that it can be more easily observed. For example, a simulation of a chemical or nuclear reaction.

EXPENSE AVOIDANCE: Many real world systems cannot reasonably be duplicated for the purpose of allowing students to observe them. A prime example is the use of flight simulators to train pilots. Some of these simulators are so complete and realistic that pilots may take all of their training on a simulator.

DANGER AVOIDANCE: It is not reasonable to allow students to freely manipulate real world systems that might be dangerous to themselves and/or others. However, this free manipulation may be essential to their learning, and it is possible with properly designed simulations. A good example is the Emergency Medical Simulator at the University of Alberta. Here, medical students have the opportunity to direct what they think should happen with a patient admitted to the emergency ward of a hospital. If they order time-consuming tests without ensuring the patient's stability, if they fail

to collect all relevant information, or if they make incorrect diagnoses, the simulated patient "dies."

FEEDBACK: Most real world systems operate without providing information about how they operate to an observer. In educational settings, this feedback about the system's operation is an essential part of the learning experience. Simulations of real world systems can have built-in feedback generation facilities included so that rapid appropriate system feedback can be produced whenever pedagogic considerations demand it. In this and other ways, simulations of real world systems can be instructionally more useful than real experience with the system itself.

Besides having the "reality modulating" capabilities outlined above, simulation CAI programs are usually intrinsically motivational. That is to say, they are usually FUN to play with. Furthermore, there are many topics in most educational settings which are amenable to instruction via CAI Simulation programs.

There is only one (obvious) disadvantage to CAI Simulation programs: they are normally quite difficult to develop. Production of a Simulation routine requires thorough research to determine exactly how the real world system operates. This system must then be represented in the form of equations, tables, heuristics, and algorithms so that it can be modelled correctly on the computer. The simulation must be made to operate in a manner that is educationally sound, with appropriate feedback indicators provided. All of this can result in a computer program that is quite difficult to develop (because of both its extent and its complexity).

Simulation programs are normally most useful in teaching students broad concepts or principles. They are usually motivational and fun. Simulations should be used where the instruction is normally difficult to do otherwise, and where high motivation is required.

Other Types of CAI

Drill and Practice, Tutorial, and Simulation are the main three types of CAI. However various other sub-classes of CAI are often reported. The two most common of these are Lab Use and Games.

LAB USE: This term is used to describe a use of computers analogous to the use of (say) microscopes in a science class. The students use a prepared program to investigate or develop results appropriate to the current lesson. This type of use is considered under Data Management and Prototypes elsewhere in this paper.

GAMES: With microcomputers, the term "games" has come to have a quite specific (and non-instructional) meaning. In current usage, games involve a "Simulation" that has no particular educational value, or an interaction where the computer mediates and resolves the activities of more than one "player." While any kind of CAI can (and should attempt to) have the motivation and FUN associated with GAMES, the term is no longer useful for the purpose of distinguishing types of CAI.

2) COMPUTER MANAGED INSTRUCTION

Computer Managed Instruction (CMI) is another term which has lost value in distinguishing a use of computers. In the early 1970s, it meant:

> "an overall system for educational management in which detailed student information, complete curriculum data and information on available resources are integrated to develop individualized programs of instruction, revise curriculum content, provide for necessary counselling and guidance, and facilitate optimum educational resource management. (A.B. Salisbury, 1971)

In the mid-1970s, both IBM and CDC were able to assure educators that their instructional systems, IIS (nee ITS) and PLATO respectively, could "handle" CMI by changing the meaning of the term. This redefinition was taken up by other CAI system vendors, and CMI came to mean the same as CAI except that the CAI program would (from time to time) advise students to read or use some offline material, and would give tests on material not covered in the CAI lessons. The aspect of instruction management most easily handled by CAI systems is Online Testing. In the mid 1970s, the term CMI degenerated further to mean Online Testing systems which provided student feedback and directions for study, and then to mean any Online Testing system. Many people still use the term CMI in this way.

The initial term provided a label for a powerful educational idea that should not be lost. For this reason, this paper uses the term CMI with its original meaning.

A teacher charged with the instruction of a class of 25 students has a considerable management problem. These students have different backgrounds, abilities, and motivation - individual differences. It is obvious that the instruction should be tailored (optimized) to each student's needs for the best instruction to occur, but this is clearly impossible. Or at least it is impossible without an instruction management system.

Around 1969, a group called Research for Better Schools started an instructional management system called IPI (Individually Prescribed Instruction). It consisted of a large number of student lessons and worksheets that were classified along two major dimensions: difficulty and progress within the subject. All were keyed to specific behavioural objectives. Students would complete their assignments and then bring them to the teacher for evaluation. On the basis of their work, the teacher would assign new lessons (worksheets) on the same topic at a higher level of difficulty, or on a new topic. By means of charts, teachers kept track of each student's progress on each of the objectives of the course. Using this system, teachers spent nearly all of their time evaluating worksheets, filling out progress charts, and prescribing new instruction. They spent almost no time teaching. However, it did have the advantage of a superior recognition of student individual differences.

Computer systems can be used to mark worksheets, keep progress charts, and dispense prescribed lessons. An IPI system that in-

corporated a computer to do these clerical tasks would be a minimum CMI system. To be complete, it would be necessary for the system to also:

1) Keep diagnostic information on the curriculum materials. For example, curriculum materials that (on the average) failed to result in an improvement in student progress would be marked for revision.

2) Monitor student progress according to teacher-specified individual criteria. Reports would automatically be produced for students who did not maintain (or significantly exceeded) the rate of progress their teacher expected.

3) Keep track of the availability and use of educational resources. Expendable materials (worksheets) and re-usable materials (filmstrips, textbooks, lab kits) would be managed so that supplies would be maintained and resources were always available when students needed to use them.

4) Provide the ability for teachers to easily retrieve all information concerning individuals and groups of students, and the ability for teachers to override default student management and direct special lessons and management for individuals or groups of students.

Such systems are not likely to be available on microcomputers for some time because of the volume of online information that would be needed.

3) ONLINE TESTING

While full CMI systems cannot be provided on microcomputers, the Online Testing application is more feasible with micros than it is with larger computers. The term Online Testing means that students use the computer to take tests. Questions can be displayed, one by one, with the student answering each before he goes on to the next (interactive); or questions can be displayed in groups with the student determining the answers away from the computer and then coming back to enter the answers later (batched). In either case, the computer must keep records of the questions asked, the answers given, and the results for the teacher; and on the basis of the student's answers the computer may provide directions for further study (remediation).

The online testing system may be general, provided for a certain topic and giving all of its questions to each student who takes the test; or it could be selective, giving questions to students on the basis of past student performance, teacher direction, or other criteria. It could be linear, giving every student the questions in the same order; randomized, giving questions in a random order; or adaptive, choosing each question on the basis of the student's answers to previous questions. Adaptive systems must be interactive and selective, but all eight other combinations are possible.

General online testing programs are the easiest to prepare. The teacher knows the content which is to be covered, and can make up questions to assess student learning in the usual way. These que-

stions along with their correct answers can be programmed into a microcomputer for presentation to the student. Whether the student answers one question at a time or gets them all on a printout is a function of the type of questions being asked and teacher preference. There is little difference in programming difficulty. With a little extra effort, the question order can be randomized in either the interactive or batched mode of delivery. However, since each test has a specific content and audience, its range of use is restricted. While a group of teachers might get together to design a test for their own use, it is unlikely that such a testing program would be useful to a large number of teachers. Even the teachers who originally designed it would probably want to revise it for each new group of students to use it.

Selective tests are more difficult to develop but have a broader range of use. A stand-alone program can be made to select particular questions to be given to students on specified criteria, but the usual approach is to set up a more general program which reads in questions from a test item bank. Two programs are needed: one to allow teachers to add, replace, edit, or delete items so that the bank can be tailored to the needs of individuals or groups of teachers; and a second program to deliver the selected items to students. Items in the bank can cover more than one content area, and may be entered for the specific use of only a few teachers. The item banks can be built up and improved over several years. Teachers can trade items and thus extend the range of questions available in their testing program. One such program can be used to manage different test banks for any number of topics and/or specific teacher groups. The program that delivers the test items to the students can be made to select items linearly or randomly, and produce them interactively or in batch.

Interactive item bank systems are generally restricted to True or False, Multiple Choice, Matching, or Numeric Value Answer questions. Batch item bank systems however, can present questions of any form (including short answer and essay questions), but these must then be marked by the teacher. If the system is to keep student records, then the teacher must enter the results of the marking into the system.

A test banking system is considerably more difficult to develop than a general testing program. Careful thought must be given to the means by which items will be selected so that appropriate criteria can be included with each item. Question answers must be encoded in a standard way so that the delivery program can mark student answers correctly. A general method of storing and retrieving student results must be established; and this is difficult to do over diverse content areas.

Adaptive testing systems are a quite recent invention. Using an adaptive system, test items can be selected which most closely measure a students current ability. It can avoid asking questions that are clearly too difficult or too easy for any particular student. Thus all students taking the test feel competent. It would be impossible to make a pencil and paper test to assess any student's mathematical ability. Questions that are appropriate for a grade two student are not appropriate for a grade eleven student. But an adaptive system, with two or three preliminary questions, can quickly establish the students approximate mathema-

tical ability level and then choose particular questions to assess it more precisely. However this kind of broad spectrum testing is rarely required in the public school system.

4) PRESENTATION AIDS

Many of the microcomputers can be attached to TV screens, and most schools have large monitors for use with their Video Tape Recorder systems. Computer programs can be designed to draw diagrams and display text on these devices so that teachers can develop their own presentations using a microcomputer. Programs designed and used for this purpose are called Presentation Aid programs.

Prompting Programs

In its simplest form, a Presentation Aid could display lines of text on the screen. At the press of the space bar, the computer would display the "next" page of text. In this way, the micro and the monitors can be used in the same manner as an overhead projector with prepared transparencies. It has an advantage (over an overhead projector) in that all of the pages are kept in order (they can't be shuffled or lost), and any one can be accessed very quickly. The disadvantage is that it is not easy to make impromptu additions to the display.

Such programs are quite easy to make, and can be readily modified as needed from year to year. Since the program provides only the outline of the content (and not all detail), a Presentation Aid is more likely to find acceptance by a broad group of teachers teaching similar topics. After the lesson, students could use the Prompting Program to review what they had learned.

Demonstration Programs

More advanced Presentation Aid programs can take advantage of the computer's ability to do animation and calculations. Some processes (for example in Science, Home Economics, or Health) are difficult to depict using static displays. The computer can be used to change the display quickly enough to convey progress, movement, or development over time. Constructing diagrams for specific problems can be a necessary but tedious task for teachers. Using a prepared program, the computer can generate the diagram based on data for a specific problem. The teacher can then show the students many more examples than would otherwise be possible. The teacher can also use such programs to illustrate the changes in the diagram that occur with changes in data. After the lesson, the Demonstration Program could be used by individuals or groups of students to do laboratory activities developed around the program's capabilities.

5) PROTOTYPES

With the availability of microcomputers in schools, it is now possible for students to get practice and learn to use the very systems that are used by businesses. Only a small percentage of students graduate from high school and take significant post secondary education. Most of them graduate and go immediately into the work force. While students can learn some "marketable" skills in school (such as typing, and basic accounting), the availability of microcomputers will increase the number and scope of potential

marketable-skill development courses in schools.

The main subject areas that will be initially affected are Business and Computing. It is already feasible for a student to learn enough about computing in high school to be able to accept a job as a programmer (on a microcomputer) upon graduation. Clerical people who have training on a Word Processing system are in high demand, and can command impressive salaries. With a microcomputer, there is no reason why high school students cannot get such training; in fact many schools offer such programs already.

There is a wide variety of business systems in current use. While it is not possible for schools to have copies of all such systems, it is possible to have prototypes of each general kind on which students can practice. With such practice, students can appreciate, at a conceptual level, the need for and use of each business system. The use of similar systems in a business environment would then be readily learned. Students would be able to assure prospective employers that they have had experience with similar systems.

Since microcomputers are so versatile, it is possible to use one micro to emulate a number of different business systems. With this one machine, business teachers have the capability offer emulations of a wide variety of different business experiences to their students. Students would learn not only the general concept and use of particular systems, but the actual operation and use of a range of specific systems as well.

Microcrocessors are being introduced into a wide variety of consumer goods. For example, automobiles are using them to control ignition timing, spark plug firing, and carburation as well as helping the driver understand any faults the car may develop. Electric ranges, TV sets, microwave ovens, automatic washing machines, and other household appliances are starting to incorporate microprocessors. In the near future, schools will be able to use microcomputers to emulate the operation of these devices to help students understand how they work, and to provide practice in diagnosing faults. A large portion of non-academic course work could come to depend heavily on the availability of systems emulated on a microcomputer.

6) DATA MANAGEMENT

Most secondary school courses are concerned with the manipulation of data. Because data manipulation is difficult and time-consuming, the actual handling of data is rarely done at the present time. However, the availability of a microcomputer makes such data management a lot easier. There are three main ways in which school classes might want to manipulate data: through Data Analysis, Data Generation, and Data Searching and Sorting.

Data Analysis

The most common form of data analysis that a high school might undertake is to collect measurements of something of interest and then calculate averages (and other statistics) and draw representative graphs. For example, a Social Studies class might make up a questionnaire on political preference. The students could then

give the questionnaire to their parents and friends, and then analyse the results. A Home Economics class might gather data on height, weight, and personal measurements and make correlations with clothing sizes; or they could collect data on normal meals eaten, calculate the nutritional value (on the average) and compare with accepted standards. An English Composition class might examine "good" prose to determine the most common types of sentences (simple, complex, compound), or the average frequency of prepositional phrases.

Using contemporary local data makes the lesson more "real" for the students, increasing motivation. It gives them a better insight into what is meant by averages (and other statistics) and how these are determined. With a microcomputer, lessons such as this can be completed in only one period (with data collection done out-of-class), making the process completely feasible for most topics. Programs to accept data, calculate statistics, and print graphs and charts are readily available so that no development would be required.

Data Generation

Most microcomputers are able to generate (pseudo) random numbers. With programming, this facility can be used to generate values according to any distribution (such as Normal, Square, Binomial). Generated data of this kind can be used to study both probability and statistics. Attachments are available to enable the microcomputer to sense such things as changes in temperature and pressure. With these, the micro can collect measurements over time which can be used to study phenomena.

While this is a use of microcomputers that cannot be used in so many subjects, it has all of the advantages discussed above (under Data Analysis).

Data Searching and Sorting

Once data has been collected or generated, the microcomputer can be used to search through the data for patterns or specific values. In one example, a Social Studies class obtained the (detailed) results of a survey done by the government on voting in the last election. The data included information on the percentage of votes cast for each party in each poll, and demographic data about the voters in each area. A Computing Studies student developed a program for them which could test hypotheses by searching through the data and sorting the results. For example, one might suppose that rural communities voted socialist. The program would find all of the rural polls and report the average vote distribution. Other hypotheses such as "City people vote conservative", "People in lumbering communities always vote against the incumbent", and "People in the colder parts of the province probably didn't cast as many ballots" were testable with this system.

7) UTILITY PROGRAMS

In addition to the directly instructional uses reported above, there are also uses that provide classroom support for the teacher. For example, using the random number generating capability, a program could be prepared which would generate worksheets of various kinds. A sheet of twenty five subtraction problems where the minuend is a two-digit number, the subtrahend is a one-digit number, and borrowing is necessary, would be tedious for a teacher to make. But a microcomputer could turn out hundreds of such sheets (all different if this was desired) with little difficulty. It could also make an answer key (if programmed to do so). Similarly, the micro could be programmed to turn out worksheets involving dates to be learned, simultaneous equations to be solved, or chemical equations to be balanced.

Another tool that the teacher might find useful is a program to make up schedules. For example, a Social Studies teacher might use such a program to randomly group students to choose their data collection assignments. In a Science class, students could be grouped and assigned to work stations and laboratory activities in a manner that would optimize the use of the facilities, and at the same time would be completely fair (unbiased).

The microcomputer could also be used for marking student assignments and calculating class averages and other statistics. If an optical card reader was available, students could code their (true/false, multiple choice, or numeric value) answers to worksheet problems or exams on cards. The computer could then do all of the rest and print out a report of the results.

The applications of Computer Assisted and Computer Managed Instruction have been implemented and studied for some time on main frame computers. The other instructional applications listed here (Presentation Aids, Prototypes, Data Management, and Utility Programs) have not really been feasible for the classroom teacher up until now. The description and discussion of these applications is shorter because not much is known about them yet. With the rapidly expanding availability of microcomputers in the schools of North America, it will be only a short time until each of these has been explored and extended (or abandoned). The microcomputer provides new means for teaching old subjects, and new topics to be learned. The changes to the business of education have just begun.

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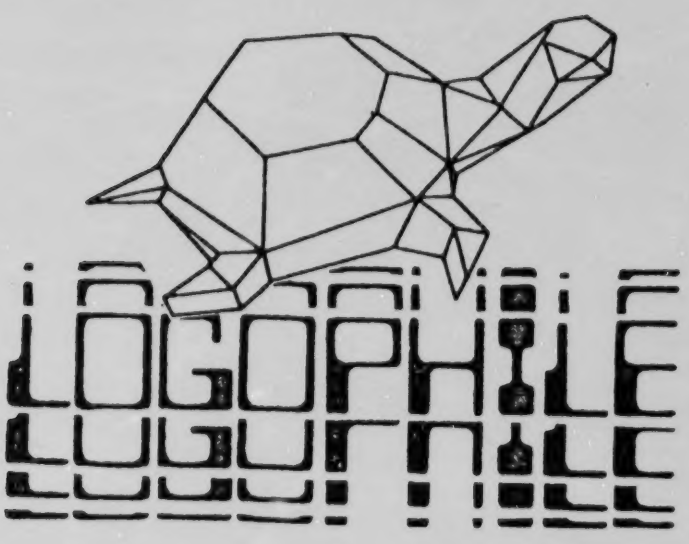

When we were little.....we went to school in the sea. The master was an old Turtle - we used to call him Tortoise. "Why did you call him Tortoise if he wasn't one?" Alice asked. "We called him Tortoise because he taught us," said the Mock Turtle angrily. "Really, you are very dense."

- Alice's Adventures in Wonderland.

LOGOPHILE 1

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LOGOPHILE 1

ECOO/SIGL and Alternatives

The Special Interest Group LOGO of ECOO was started in May 1981 at the Annual Conference. A number of ECOO members, stimulated by Seymour Papert's book (Mindstorms: Children Computers and Powerful Ideas, New York: Basic, 1980) and keynote lecture at the conference, decided to form a loose organization dedicated to the collection and dissemination of information about LOGO materials and activities. We are, as yet, an unofficial SIG. The ECOO Board will be approached shortly with a request that SIGL join SIGEE (Elementary Educators) as an official interest group of ECOO. The purpose of this first newsletter is to bring interested parties up-to-date with respect to LOGO developments over the past six months and to inform them about some forthcoming events.

An early problem for our group concerns our name. Bureaucratically correct as it may be, ECOO/SIGL sounds more like a sneeze than the rallying cry for a group of innovative educators. LUGO is a standard alternative but it too is less than mellifluous and is, at least in the short term, inaccurate; perhaps WLTBLU-O, We'd Like To Be LOGO Users Ontario? In our bilingual province TORTUE has a nice ring (and a vision of the third R changing from torture?); while the Bloor Street contender TortOISE can expect a cool reception in the provinces (en francais one gets the right idea from the beginning). A late suggestion from our musical division is LOGO Action Research Group Ontario; a possible handle with immediate images of turtlely torpor? These and other possibilities will be discussed at our November 21st meeting at the Ontario College of Art (see following item) so come along or send us your ideas if you are unable to attend.

LOGO Availability

The major frustration for would-be LOGO users is accessibility. Most people find it difficult to get any hands-on experience with the language. So some observations on accessibility as we understand it at the end of October.
Although we know you've heard this song before, this situation is, we hope, only of short duration. At the moment there are three microbased versions of LOGO. The only one of these which is commercially available, at least in theory, at the time of writing is the Texas Instrument system. The good news here is about the LOGO part of the system. It features not only a torpid turtle but also a 32-tet of spirited 'sprites' which make graphic life very interesting indeed. The bad news (how shall we put this. ..not wishing to be sued in our very first number) has to do with the host machine. It would not be an exaggeration to say that the 99/4 has been less of a success than TI shareholders might have wished. The net effect of this is that this version, at least in Ontario, is effectively unavailable at this time.

The other two versions, neither of which have yet been released, are based on the Apple microcomputer. One of these has been developed at MIT by Professor Abelson and his group. Following lengthy negotiations between MIT and the National Science Foundation, which has funded the LOGO Project for a decade, we understand that this version needs only a distribution arrangement before it becomes commercially available. Companies likely to be in the running here include Microsoft, Apple and Terrapin. The rumour that TI had attempted to impede the release of this version of LOGO is apparently without foundation. Two Ontario universities, Queen's and York, are among approximately 25 approved test-sites for this system. If all proceeds smoothly from here, MIT Apple LOGO might be available before Christmas. The minimum configuration for running this version will be a 48K Apple with a single disk drive and a 16K language (memory expansion) board. No firm costing figures are known but $200± for the disk seems a possible figure.

A second Apple-based system is being developed by LOGO Computer Systems Incorporated, a company with headquarters in Montreal and another office in Boston. (Many ECOO members will remember Cynthia Solomon, Margaret Minksy and Jim Baroux of LCSI from their fine contributions to the 1981 conference.) LCSI hopes to have a package ready before Christmas which will include a 16K expansion board, a LOGO disk and a copy of Mindstorms for something in the neighbourhood $300±. They are also developing a 64K board to go on top of their first system. This board will, among other things, make sprites available. LCSI, looking to the future, hopes to develop Atari, Radio Shack Colour and IBM Personal versions of LOGO. They intend to offer courses and support materials as well and have plans for walk-in LOGO Centres. The first of these is due to open soon in New York City.

LOGO Ontario Activities

Past

In July, Dr. Sterling Beckwith of York University, the doyen of Ontario LOGOers, gave a stimulating lecture/workshop at Queen's entitled "LOGO vs. BACH".

In late October one of a series of workshops offered at the Ontario Science Centre by TERC (Technical Education Research Centers of Cambridge, Mass.) was on LOGO. An introduction to the principles and pedagogy of LOGO was followed by a hands-on session using TI equipment.

Ongoing

A fall term graduate seminar on LOGO is underway at Queen's. Among the participants are instructors from the Faculty of Education, students in the Master of Education programme and personnel from Kingston area boards of education.

Upcoming

On Saturday, 21 November there will be a meeting of ECOO/SIGL from 4pm to 6pm at the Ontario College of Art, 100 McCaul Street, Toronto. This meeting will be part of the week long (16-22 November) 'Computer Culture '81' conference held at the OCA and the Art Gallery of Ontario. We hope to have Brian Silverman give us a demonstration of the LCSI Apple LOGO. An ECOO/SIGL Chinese Dinner will follow the session for interested participants.
Information about 'Computer Culture '81' is available from 'The Photo/ Electric Arts Foundation' Box 7109, Postal Station A, Toronto M5W 1X8. Details about the LOGO session will be available Saturday at the main registration desk or from Dr. Sterling Beckwith at Winters College, York University (416-667-3217).

The dates for the 1982 ECOO conference are 28 April - 1 May. It is hoped that some LOGO sessions will find their way onto the programme.

There are tentative plans for some short inservice sessions on LOGO at Queen's. Depending on interest, one-day sessions might be scheduled in February, April, June and July. Interested parties are asked to write or telephone Dr. Dale Burnett (613-547-6971) or Dr. William Higginson (613-547-6162) at the Faculty of Education, Queen's University, Kingston, Ontario K7L 3N6. (Hardware availability will limit the number of participants in any session. Please indicate if you can bring a 48K Apple with a language board.)

It would seem a good idea to have some sort of ECOO/SIGL Summer Workshop. Possibilities will be discussed at the 21 November meeting. Let us know your ideas.

Plans for the CEM (Canadian Educational Microcomputer - known in some quarters as the 'Bionic Beaver') are rumoured to include the capacity for LOGO.

BibLiOGOgraphy

Turtle Geometry

The major LOGO news of the summer was the publication of "Turtle Geometry:The Computer as a Medium for Exploring Mathematics" by Harold Abelson and Andrea diSessa (MIT Press, 1981, 0262010631, $20 US). A detailed review would be appropriate for a future number of LOGOPHILE but at this point let us just note that this book is 477 pages of cerebral dynamite. The familiar triangles, squares and 'houses' of LOGO 001 are there, in the first few pages at least, but eight chapters later one finds oneself "At the Edge of a Wedge" in "Curved Space and General Relativity". Just the reference you need for those discussions with your colleagues which start, "LOGO,...well the pictures are very pretty, but...".

Journal Articles

A number of popular computing journals have published articles on LOGO. Among the most interesting are three written by Harold Nelson, 'LOGO for Personal Computers' Byte, June 1981; 'Seymour Papert: Spearheading the Computer Revolution' and 'Learning with LOGO', both in the Summer 1981 issue of On Computing.
Mindstorms has been reviewed widely if somewhat superficially in a number of places. One attempt at a critical analysis is the 'Whither Goes the Turtle' paper by Rousseau and Smith in the September 1981 number of Kilobaud Microcomputing (#57). In this same issue there are papers by six teachers from the Dallas Lamplighter School, 'Learning with LOGO'; Richard Carter, 'LOGO and the Great Debate' and an excellent article by Sylvia Weir, 'LOGO and the Exceptional Child' which clearly shows the potential of LOGO for Special Education.

Contributions to LOGOPHILE 2

It would seem a good idea to have a second issue of LOGOPHILE just before the 1982 conference. Accordingly, the editor would appreciate receiving names, notes, notices, references, programs, articles, print-outs,letters,... by the end of March. Should you wish a copy of LOGOPHILE 2, please send a self-addressed, stamped, 81/2 X 11 envelope to:

Dr. William Higginson
Editor, LOGOPHILE
Faculty of Education
Queen's University
Kingston, Ontario K7L 3N6.

APPENDIX C

CONTRIBUTIONS FROM

WORKING GROUP C

Working Group C AGENDA POSSIBILITIES Paper # 1

Our task for the June, 1981, meeting will be first, to diagnose the state of Calculus teaching in terms of issues which appear to need to be resolved, and second, to begin the process of discussing their resolution, one at a time. It is expected that there will be considerable carry-over into future meetings.

The following issues come readily to mind.

1. Is Calculus the same as Introductory Analysis?

   This issue arises mainly at the university level. It is closely linked to 2. (The foci are different. A recent trend has been to replace a good introduction to Calculus by a premature, heavy-handed and under-developed introduction to Analysis.)

2. How rigorous should one be, and when?

   Teachers often find themselves on a tight-rope in this regard. They must not aim at complete proofs, yet our high schools are sometimes criticised for using too many formulas which are simply given, without any justification. This problem is most acute with Calculus, rather than Algebra.

3. What should be done about differentials?

   From many textbooks one would gather that differentials (whatever they may be) play a key role in, for example, integration by parts, the meaning of integration generally, the Chain Rule and the Fundamental Theorem. On the other hand, many professors (at times heatedly) deny this, and rigorously exclude them. This can be confusing. (Omit them entirely. They are artificial, misleading and their manipulative convenience is greatly overestimated.)

4. How much emphasis should be given to "funny" functions - for example, those which use absolute values, or other means, to produce apparently contrived discontinuities? (This is tricky. Functions which are not everywhere continuous or not everywhere differentiable should be discussed, but a thorough treatment takes a lot of time and results in a false emphasis.)

5. How should one sketch polar graphs?

   The standard text-book method is to draw up a table of values. Yet most mathematicians don't do this. They first sketch the (usually familiar) Cartesian graph, and then convert to the polar representation from this. (A minor topic, but an important issue. Students are drilled in Cartesian graphs, but when an opportunity to use them (in Polar graphing) arises, it is overlooked. Why do we make a fuss over certain things in some places and ignore them elsewhere.)

6. Can the usual first-course (single-variable) text of some 600 to 800 pages be made to yield a more compact version of the central ideas?

   One would certainly hope so, yet mammoth texts continue (in North America) to dominate the market.

7. Should there be more uniformity of notation?

   $\frac{dy}{dx}$ versus $\frac{Dy}{x}$ (We should either try for more uniformity among courses or make a point of using all the standard notations in one course.)

8. Which applications should be included?

   What are the criteria? What purposes should be served? Do any have a special status - e.g. should any be recommended for inclusion in all introductory Calculus courses? (The terminology of "applications" should be replaced by that of "interactions", and the matter of motivation for studying applications or interactions rates more thought than is usually given to it.)

9. What should the division of labour between High School Calculus and University Introductory Calculus be?

Should the high schools keep exclusively to rational functions? What overlapping, if any, is desirable or unavoidable?

10. Should techniques of integration be treated as a unit or interleaved with applications as various classes of functions are met?

(The interleaved approach gets quite messy and has artificial gaps. A systematic approach has strong internal motivation and is more efficient.)

11. How should $\ln x$ be introduced?

Defining it out of $1/x$ has been very popular, but some are now having second thoughts.

12. Should (simple) differential equations play a stronger role?

13. What role should historical matters play?

(A stronger role than is provided by one-paragraph biographical inserts. Every Calculus course for which it is within reach should give a derivation of Kepler's Laws from Newton's Laws and have something to say about the Scientific Revolution.)

14. Are there "best" or "better" proofs of basic theorems?

e.g. Chain Rule. Fundamental Theorem. Mean Value Theorem. (Intuitive simplicity is currently being ruthlessly sacrificed for explicitness and completeness with respect to special cases.)

15. Epsilonics.

How much? When? How? Connection with approximation theory? Use of inf, sup, reduction to monotone functions. (Exercises in calculation the "best" δ for given ε can be very misleading. In Calculus generally, outside of Approximation Theory, only the existence of δ, or bounds for it, are relevant.)

Some work has already begun on 8 (1980). At the June meeting comments will be made on each of 1-15, and a selection of items for detailed study will be recommended. These will include 1•, 2•, 3•, 9•, as well as a follow-up on 8.

Logarentials

(hypothetical text presentation, abbreviated)

Notation

If $a^x = b$ then we write $x = \frac{\ell b}{\ell a}$, read "ell b by ell a".

The bar does not mean division. (At least, not yet.)

Definition of the logarential

If t is the independent variable, then $\ell t = 1$. For y generally, $\ell y = (\frac{\ell y}{\ell t})\ell t$ (Note: as above, "$(\frac{\ell y}{\ell t})$" isn't (yet) a quotient, even though it looks like one.)

Theorem

Generally, $(\frac{\ell y}{\ell x}) = \frac{\ell y}{\ell x}$ where the left side isn't a quotient, but the right side is the quotient: ℓy divided by ℓx

Proof: the right side quotient is

$$(\frac{\ell y}{\ell t})\ell t \Big/ (\frac{\ell x}{\ell t})\ell t = \frac{(\frac{\ell y}{\ell t})}{(\frac{\ell x}{\ell t})} \quad (\rightarrow \text{ a quotient})$$

(where the bracketed terms are not quotients)

But, by the Conversion Formula, this is

$(\frac{\ell y}{\ell x})$ (not a quotient), which is the left side. q.e.d.

Consequence 1. The Conversion Formula now takes the form of a cancellation.

$$\frac{\ell a}{\ell c} = \frac{\ell a}{\ell b} \cdot \frac{\ell b}{\ell c} \quad \text{(all three are quotients)}$$

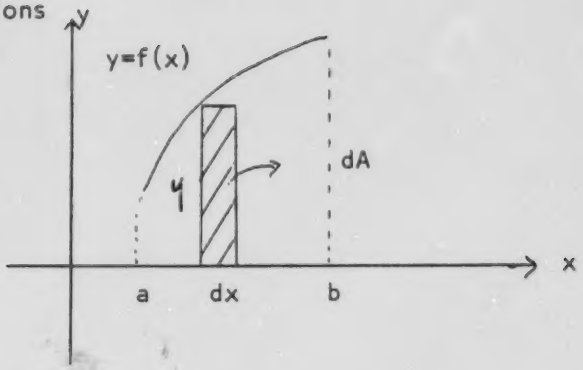
This is why (in the year 200x) we write logarithms in logarential form.

Consequence 2. The choice of independent variable doesn't matter.

Setting Up Integrals

2
Presentations

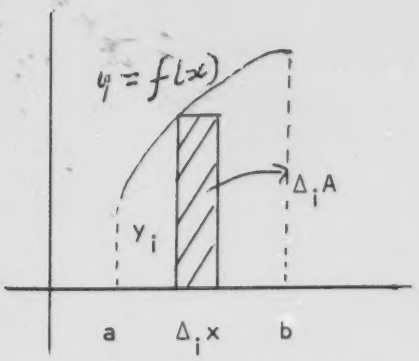
I



$$\text{Area} = \int_a^b dA$$

$$= \int_a^b y\,dx$$

II



$$\text{Area} = \lim \Sigma\, \Delta_i A$$

$$= \lim \Sigma\, y_i \Delta_i x$$

$$= \int_a^b y\,dx$$

Imperfection by omission

versus

Imperfection by confusion

By Parts

Three presentations

I. $d(uv) = udv+vdu$

Apply $\int$ to both sides (?)

$uv = \int udv + \int vdu$

$\int udv = uv - \int vdu$

II. $D_x(uv) = uD_xv + vD_xu$

$uv = D_x^{-1}(uD_xv)+D_x^{-1}(vD_xu)$

$D_x^{-1}(uD_xv) = uv - D_x^{-1}(vD_xu)$

III. $D_x(uv) = uD_xv + vD_xu$

$uv = \int uD_xvdx + \int vD_xudx$ (1)

$= \int udv + \int vdu$ (Chain Rule)

$= \int udv + \int vdu$

Hence $\int udv = uv - \int vdu$

or (from (1), directly)

$\int uD_xvdx = uv - \int vD_xu\,dx$

(du, dv don't appear at all)

(nor is the Chain Rule involved)

Modes of calculation

1. $\int x\ell nxdx$ [Set $\ell nx = u$, $xdx = dv$

$d\frac{x^2}{2} = dv$]

$= \int udv = uv - \int vdu$

$= \ell nx \cdot \frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x}dx$

$= \ell nx \cdot \frac{x^2}{2} - \frac{x^2}{4} + c$

2. $\int x\ell nxdx = \int \ell nxd\,\frac{x^2}{2}$

$= \ell nx \cdot \frac{x^2}{2} - \int \frac{x^2}{2} d\ell nx$

$= \ell nx \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$

$= \ell nx \cdot \frac{x^2}{2} - \frac{x^2}{4} + c$

3. $\int x\ell nxdx = \int \ell nx(\frac{x^2}{2})' dx$

$= \ell nx(\frac{x^2}{2}) - \int \frac{x^2}{2}(\ell nx)' dx$

$= \ell nx \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$

$= \ell nx \cdot \frac{x^2}{2} - \frac{x^2}{4} + c$

Paper # 5

THE LEIBNIZ NOTATION

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1. Elementary calculus uses two notations: one traditional and based on the notation used by Leibniz, the other more modern. Rigorously-proved theorems in the differential calculus use the modern notation (no-one today ever quotes the mean-value theorem in Leibniz's notation) and many analysis texts use $\int_\alpha^\beta f$ to denote the integral of f from α to β. In the less theoretical areas, however, the Leibniz-type notation is still much used. Its advantages are well known: among them are the very familiarity of dy/dx and $\int f(x).dx$, the neat chain-rule, and the ease of integration by substitution. Moreover, in applications integrals seldom or never appear without differentials; indeed in physics there is a vital distinction between

$$\int F.ds$$

and

$$\int F.dt$$

(the integrals of force with respect to distance and with respect to time).

However, there are difficulties with Leibniz's notation. The usage in physics that we have just mentioned is in sharp contrast with the traditional usage in elementary calculus, in which

$$\int_\alpha^\beta f(x).dx$$

is not merely equal to, but is actually the same as

$$\int_\alpha^\beta f(u).du$$

each being the integral of f from α to β, the x and the u being essentially meaningless. Moreover, in the traditional usage,

$$\int_\alpha^\beta y.dx$$

is not defined unless y is a function of x, whereas physicists need $\int F.ds$ whether or not F is a function of s. Physicists also need

$$\int r^2.dm$$

for the moment of inertia of a wire. Here a definition which requires the integrand to be a function of the variable of integration raises real difficulties; it is not clear whether it even makes sense to ask "is r^2 a function of m ?"

There are also purely logical difficulties. If

$$u.dv = s.dt$$

it does not, in the traditional notation, follow that

$$\int_\alpha^\beta u.dv = \int_\alpha^\beta s.dt \ .$$

This disagrees not only with common sense, which suggests that we should be able to replace equals by equals, but with the standard modern advanced-level treatment of differentials: if $u.dv$ and $s.dt$ are two equal Fréchet differential forms, it does follow that

$$\int_C u.dv = \int_C s.dt.$$

Finally, the commonest difficulty that students find with the usual presentation of the traditional notation is the following. If

$y = f(x)$, then $dy/dx = f'(x)$; if, say, 2 is in the domain of f' , how can it make sense to write $f'(2)$ but not make sense to write $dy/d2$?

It is the purpose of this note to suggest how these difficulties might be overcome.

2. The reason for the last difficulty is not hard to find. Definitions of differential (or of derivative in Leibniz-type notation) do not always make clear what types of mathematical entities are denoted by the various letters used. By contrast, a typical definition of f'(x), such as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

makes it quite clear that f and f' denote functions, while x denotes a number. If we want to be formal, we can say that the symbol f and the compound symbol f' are function-variables, while the symbol x is a numerical variable. Because students are so familiar with the equation

(i) $$dy/dx = f'(x)$$

they may jump to the conclusion that the x in dy/dx is also a numerical variable. It is, however, a function-variable; that is why it cannot be replaced by a numeral.

The two x's in (i) must denote the same thing; therefore the question arises "what does f(x) denote if f and x both denote functions?". The answer is that it denotes the composite of f with x. Thus f(x) denotes two different things according to the meanings of f and x. If f denotes a function and x a number in its domain, then f(x) denotes the value of f at x; if f and x denote functions, then f(x) denotes their composite. (The double meaning for f(x) can be avoided by the use of the modern notation for the composite, but only at the expense of turning the familiar "If $y = f(x)$ then $dy/dx = f'(x)$" into the unfamiliar "If $y = f \circ x$, then $dy/dx = f' \circ x$".)

3. Let us take a hint from the usual treatment of the real-number system. Here the axioms of a complete ordered field are taken as fundamental properties. They are proved from some definition of real number and, once they are established, the properties of real numbers needed in analysis are developed from them rather than directly from Dedekind sections, equivalence-classes of Cauchy sequences of rational numbers, or whatever. For example, the theorem that if r is any real number there is a integral real number n greater than r is always proved from the completeness-property, although it would be simpler to exhibit the Dedekind section n directly.

Let us, then, start with a set of fundamental properties for the differential. We shall need two: If x and y are functions and α and β are numbers

(I) $$dy/dx = y'/x'$$

and

(II) $$\int_\alpha^\beta y.dx = \int_\alpha^\beta y.x' \ .$$

(We are of course using the modern notation for integrals, so the right-hand-side of (II) is the integral of the function $y.x'$ from α to β.)

How will these properties be presented? Extant calculus texts vary in style and level, and these properties can be presented in correspondingly varied ways. Some texts introduce dy/dx as another notation for $f'(x)$, where $y = f(x)$. In this style we can introduce dy/dx as another notation for y'/x'. Other texts define differentials; for example they define dx to be the function for which $dx(\alpha,\tau) = x'(\alpha).\tau$. From their definition

they prove that if $y = f(x)$ then $dy/dx = f'(x)$; in this style we would prove from the definition that $dy/dx = y'/x'$.

All extant texts define $\int_\alpha^\beta y.dx$ to be the integral of f from α to β if y is of the form $f(x)$ and leave it undefined if y is not of this form; in place of this, we would define $\int_\alpha^\beta y.dx$ to be the integral of $y.x'$ from α to β , noting that if $y.dx = u.dv$ then $y.x' = u.v'$, and that consequently our definition is valid: $\int_\alpha^\beta y.x'$ is uniquely determined by α , β , y and x .

Of course, if we take our differentials to be Fréchet differentials (in one dimension)and if we interpret $\int_\alpha^\beta y.dx$ as the integral of the differential form $y.dx$ over the interval $[\alpha,\beta]$, we obtain properties I and II.

4. Let us see how Leibnizian-style calculus can be developed from the two fundamental properties.

The standard theorems about differentials of sums, products, etc., follow easily. For example, from the theorem "if u and v are differentiable and $w = u+v$, then $w' = u'+v'$" we can prove the corresponding theorem about differentials as follows.

Theorem

If u and v are differentiable and $w = u+v$, then $dw = du+dv$.

Proof. Let t be any function whose derivative is everywhere non-zero, for example, the identity-function. Then successively

$$w' = u'+v'$$

$$w'/t' = u'/t'+v'/t'$$

$$dw/dt = du/dt+dv/dt$$

$$dw = du+dv .$$

The theorem about the differential of a function of a function appears (for differentiable f and x) as

$$\text{if } y = f(x) \text{ then } dy = f'(x).dx.$$

(In Bourbaki's notation this would be "if $y = f\circ x$ then $dy = f'\circ x.dx$.")

The chain-rule is

$$\frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx}$$

at any point at which the right-hand-side exists. (Here I am of course using the familiar figure of speech in which "f exists at α" means "$f(\alpha)$ exists" and "$f = g$ at α" means "$f(\alpha) = g(\alpha)$".)

Now let us turn to applications, and first to rates of change. Given a quantity that changes as time passes, we can let, say, $u(\tau)$ be the value of the quantity at time τ, for each τ. The function u then represents the variable quantity. If another variable quantity is represented by a function v, we might ask what function represents the rate of change of the second variable with respect to the first. The average rate of change of the second variable with respect to the first between times τ and $\tau + \delta$ is

$$\frac{v(\tau + \delta) - v(\tau)}{u(\tau + \delta) - u(\tau)}$$

and the actual rate of change at time τ is the limit of this as δ approaches 0, i.e.

$$v'(\tau)/u'(\tau).$$

This is the value at τ of v'/u', i.e. of dv/du, and so dv/du represents the desired rate of change.

So much for quantities that vary with time. Other types of variable quantity, such as those that vary with position, can be treated similarly. We are dealing in this note with one-dimensional functions, and in this case a position-variable will be a function of position along a curve. To represent such a variable by a function u we parametrize the curve and let $u(\tau)$ be the value of the variable at the point with parameter τ, for each τ. The argument above will apply.

In general we have the result:

> If u and v represent two variables, then dv/du represents the rate of change of the second variable with respect to the first.

When we turn to integration we could say that our definitions are bound to give the right result because, in any reasonably rigorous treatment, integrals like $\int F.ds$ and $\int r^2.dm$ are defined as Stieltjes integrals, and the relation between Stieltjes and Riemann integrals is (for well-enough behaved u and v)

$$\int_\alpha^\beta u.dv = \int_\alpha^\beta u.v'$$

which is precisely our fundamental property for integrals. (This may look momentarily unfamiliar, but it is only the modern way of writing the traditional $\int_\alpha^\beta u(x).dv(x) = \int_\alpha^\beta u(x).v'(x).dx)$. But it will be more illuminating to give an illustration.

Let us find the moment of inertia about a given axis of a wire AB lying along a parametrized curve. We let $r(\tau)$ be the distance of the point P_τ with parameter τ from the axis, and $m(\tau)$ be the mass of the portion AP_τ of the wire. We are treating the elementary case, so we assume that the wire has a proper linear density everywhere: there are no point-masses of infinite density. (If there were, we could not use elementary integration but would need the Stieltjes integral.) In fact, we assume that m is continuously differentiable (and that r is continuous). Form a partition of the parameter-domain $[\alpha,\beta]$ and let $[\gamma,\delta]$ be a typical subinterval in the partition. The portion $P_\gamma P_\delta$ of the wire

has mass $m(\delta) - m(\gamma)$, which is $(\delta - \gamma).m'(\xi)$ for some ξ in $[\gamma,\delta]$. It can be regarded as lying approximately a distance $r(\xi)$ from the axis, so its moment of inertia is approximately $r^2(\xi).m'(\xi).\Delta$ where $\Delta = \delta - \gamma$. Thus the moment of inertia of the wire is approximately

$$\Sigma r^2(\xi).m'(\xi).\Delta$$

summed over all subintervals of the partition. Its exact value is the limit of this as the mesh of the partition approaches zero. By definition, this limit is $\int_\alpha^\beta r^2.m'$, and by our fundamental property for integrals this is

$$\int_\alpha^\beta r^2.dm.$$

5. There is a useful piece of notation that we can take over from physics. The work done when a gas expands is the integral of pressure with respect to volume. The work done when the pressure changes from p_1 to p_2 is written

$$\int_{p=p_1}^{p=p_2} p.dv;$$

the work done when the volume changes from v_1 to v_2 is written

$$\int_{v=v_1}^{v=v_2} p.dv.$$

This notation is used informally, but it is easy to devise a formal definition. Clearly we want $\int_{p=\alpha}^{p=\beta}$ to denote integration from a number at which p takes the value α to a number at which p takes the value β.

Definition. If p and f are functions and α and β are numbers, then

$$\int_{p=\alpha}^{p=\beta} f$$

denotes $\int_\gamma^\delta f$, where γ and δ are such that $p(\gamma) = \alpha$ and $p(\delta) = \beta$, provided that this is unambiguous, i.e., provided that the set

$$\{\int_\gamma^\delta f : p(\gamma) = \alpha \text{ and } p(\delta) = \beta\}$$

contains one and only one number.

It is of some interest that when Euler introduced limits of integration (Leibniz himself did not use them) he did so in the form

$$\int y.dx \left[\begin{matrix} \text{ad } x=b \\ \text{ab } x=a \end{matrix} \right]$$

(ab and ad being the Latin for "from" and "to").

(We might as well give now a formal definition of the corresponding derivative notation.

Definition. If f and z are functions from R to R and α is a real number, then

$$\left[f \right]_{z=\alpha}$$

denotes

$$f(\tau)$$

where τ is such that $z(\tau) = \alpha$; provided that this is unambiguous. This notation is commonly used when f is a Leibnizian derivative.)

If, in our example above about the wire, we represent arc-length from the end A by s, and if the length of the wire is λ, the moment of inertia is

$$\int_{s=0}^{s=\lambda} r^2.dm.$$

Notice that this is independent of the parametrization. If we re-parametrize the curve, the functions r, s and m will all change, but the integral will still represent the same moment of inertia, and will not alter.

Again, let a smooth curve run from (x_1, y_1) to (x_2, y_2) and be such that no line parallel to the y-axis cuts it more than once. Parametrize the curve and let $(x(\tau), y(\tau))$ be the coordinates of the point with parameter τ. Then the signed area of the region between the curve, the x-axis, and the lines $x = x_1$ and $x = x_2$ is

$$\int_{x=x_1}^{x=x_2} y.dx$$

and is independent of the parametrization chosen.

We shall return to this invariance of Liebnizian-style integration in the next section. Meanwhile, we have the following result.

Theorem

If f and x are functions from R to R and α and β are numbers, then, under the obvious conditions,

$$\int_{x=\alpha}^{x=\beta} f(x).dx = \int_{\alpha}^{\beta} f.$$

Proof. $\int_{x=\alpha}^{x=\beta} f(x).dx = \int_{\gamma}^{\delta} f(x).dx$ where $x(\gamma) = \alpha$ and $x(\delta) = \beta$

$= \int_{\gamma}^{\delta} f(x).x'$

$= \int_{x(\gamma)}^{x(\delta)} f$ by the theorem on change of variable in integrals

$= \int_{\alpha}^{\beta} f$

Note: The "obvious conditions" are the conditions that the change-of-variable theorem should apply, namely that the two integrals should exist and that x' should be continuous on some interval $[\gamma,\delta]$ for which $x(\gamma) = \alpha$ and $x(\delta) = \beta$.

6. We noticed earlier a certain invariance-property. More generally, if we have three variables of position along a curve, and if we represent them by functions u, v and t by parametrizing the curve and letting $u(\tau)$, $v(\tau)$, $t(\tau)$ be the respective values of the variables at the point with parameter τ, and if α and β are two numbers, then

$$\int_{t=\alpha}^{t=\beta} u.dv$$

is independent of the particular parametrization chosen. So, for that matter, is

$$[dv/du]_{t=\alpha}.$$

Again, if we have three quantities that vary as time passes, and we represent them by three functions u, v and t, then

$$\int_{t=\alpha}^{t=\beta} u.dv \quad \text{and} \quad [dv/du]_{t=\alpha}$$

are both independent of the way in which we choose to represent the variables. For example, if the variables are the distance covered by an aircraft taking off, its velocity, and time, we might say

(A) let $u(\tau)$ be the distance covered in time τ, $v(\tau)$ the velocity then, and $t(\tau)$ the time then (so that $t(\tau) = \tau$); or

(B) let $v(\tau)$ be the velocity when the distance covered is τ, $t(\tau)$ be the time required to cover this distance, and $u(\tau)$ be the distance itself.

In specification A, time is the underlying parameter. The use of distance as underlying parameter in B is equivalent to using the distance covered to measure time. In fact, any monotonically-changing

variable can be used to measure time, including the number of vibrations of a caesium atom, the number of swings of a pendulum, or the volume of water in a clepsydra.

The condition that t, u, v and t^*, u^*, v^* represent the same trio of variables is that there should be a monotonic function q such that

$$t^* = t(q), \quad u^* = u(q) \quad \text{and} \quad v^* = v(q)$$

(or $t^* = t \circ q$ etc. in modern notation). The direct proof that then

$$\int_{t=\alpha}^{t=\beta} u.dv = \int_{t^*=\alpha}^{t^*=\beta} u^*.dv^*$$

and

$$[dv/du]_{t=\alpha} = [dv^*/du^*]_{t^*=\alpha}$$

is easy.

A physicist will usually find this invariance obvious because it follows automatically from the physical meaning of the derivatives and integrals.

Hugh Thurston
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Canadian Mathematics Education Study Group
Edmonton, June 5-June 9, 1981

Workshop: Teaching of Calculus

(Remarks made by H.N. Gupta, University of Regina)

June 6, 1981: Exponential and Logarithmic Functions

Dr. R. Christian (Vancouver) demonstrated a rather pleasant way of introducing derivatives of the exponential function a^x. In essence, this amounted to showing that $\frac{d}{dx} a^x = a^x (\frac{d}{dx} a^x)_{x=0}$. Denoting $(\frac{d}{dx} a^x)_{x=0}$ by $\ell(a)$, Christian showed that $\ell(a)$ has the characteristic properties of a logarithmic function.

Gupta observed that this was also the way in which Augustus De Morgan (1806-1871) treated the subject in his Differential and Integral Calculus, London, 1842 (p.56-57). In his preface, De Morgan cites this treatment as 'not being indebted to his knowledge of what has been written before him'. [De Morgan was vastly well-read in Continental literature on the subject]. De Morgan tentatively identifies $\ell(a)$ with $C \log a$, where the constant C depends on the base of logarithm adopted. In a later Chapter, De Morgan obtains the result that if the base of logarithms be chosen to be Naperian, then C=1 (p.75-76). For De Morgan, a^x, e^x, $\log x$ are all pre-existing functions, only their derivatives are under investigation.

Gupta gave a brief historical account of the subject of logarithmic and exponential functions. Though invented for computational purposes, the first logarithms (1614) of

Napier (1550-1617) had a simple mechanical foundation and explanation. Napier proceeded from a relationship between two continuously varying quantities x, and y, such that (in modern terminology)

$$\frac{dy}{dx} = -\frac{1}{10^7}y$$

with the added stipulation that at x=0, $y=10^7$. He called the variable quantity x as the logarithm of y. Napier's original logarithms did not even satisfy the now common logarithmic law, log(ab)=log a+log b. They, however, satisfied a prosthaphairesic law, namely, that if y_1, y_2, y_3,... are in a geometric progression, then x_1, x_2, x_3... are in an arithmetic progression. The observation that the area of the region bounded by a hyperbola, one asymptote and two lines parallel to other asymptote behaves 'logarithmically' is traceable to Gregory of St. Vincent (1584-1667) in one of his works (1647). Mercator (1620-1687) in his celebrated Logarithmotechnica (1668) discovered the area under the hyperbola $y = \frac{1}{1+t}$ from t=0 to t=x in the form of the series $x - \frac{x^2}{2} + \frac{x^3}{3}$ ....., now known as Mercator's series. The name 'natural logarithm' seems to be due to Mengoli (1626-1686). For a long time, logarithms were handled without reference to any base. It was Euler (1707-1783) who through his famous 2-volume work 'Introductio' (published 1748) introduced this vogue of looking at logarithms as inverse to exponential functions. (If $y=a^x$, then x is $\log_a y$ to the base a). To Euler we owe the formulae connecting logarithms with

different bases. In particular, the formula $\log_b a = \frac{\log_c a}{\log_c b}$. Following Euler, logarithms came to be derived in this way from the exponentials regarded as more basic. The first text-book which proceeded with logarithms as basic, defined as $\int_1^x \frac{1}{x}\,dx$, seems to be one by M. Pasch. (Einleitung in die Differential-und-Integral Rechnung, 1882). But, then, since natural logarithms were felt to be needed in Algebraic Analysis (synonymous with the Theory of Series) in advance of a course of instruction in the Calculus, the idea of starting with logarithms as an integral did not strike deep roots until fairly recently. Even the terminology 'hyperbolic logarithms' had nearly sunk into oblivion.

June 7, 1981: Differentials

Ralph Staal (University of Waterloo), suggested that 'differentials (whatever they may be) are entirely artificial, misleading and their manipulative convenience is greatly overestimated'. Staal demonstrated by examples that the use of differentials in anti-differentiation problems is at best mnemonic and should better be dispensed with.

Gupta conceded that in formal manipulative situations, the use of differential notation is perhaps dispensable; he pleaded, however, for a greater and more effective use of differentials as tools both of approximation and of discovery. For example, the growth of most functions f(x) of familiar occurence from x=a to x=b can be approximated through summing up 'difference's of over n successive subintervals

$(a=x_0,x_1)$, $(x_1,x_2),\ldots(x_{n-1},x_n=b)$, not necessarily all equal. If the functions f(x) itself is not known but the law of its growth is somehow known or assumed to be of the form $f'(x)=g(x,y)$, g known, one may substitute for the difference $f(x_r)-f(x_{r-1})$ the expression $f'(x_{r-1})(x_r-x_{r-1})$, ie. the differential $df|_{x=x_{r-1}}$, or for that matter even the differential $df|_{x=c_r}$, where c_r is any value in the interval (x_{r-1},x_r). Although, the difference $f(x_r)-(x_{r-1})$ is in general, not equal to $f'(c_r)(x_r-x_{r-1})$, the facts of the Cauchy-Riemann theory of integration can be invoked to legitimize what may be called the 'principle of Condensation of errors': if the number n of intervals is sufficiently large, so that the width of each interval is sufficiently small, then $\sum_{r=1}^{n} f'(c_r)(x_r-x_{r-1})$ is approximately equal to f(b)-f(a). (Of course, the unknown function f(x) should be and can confidently be assumed to be sufficiently well-behaved: continuity throughout the interval [a,b] and differentiability in (a,b) would do.) For practising scientists, the power of the Calculus lies primarily in its ability to 'obtain' a function when the 'changes' associated with the function are known to be connected in a certain given manner; in other words, the real power of the Calculus for scientists consists in its power to solve differential equations. No doubt as the subject progressed, formal tricks have developed eliminating the need for students to 're-discover' solutions. The following examples may illustrate the point. Take, for instance, the equations (i) $f'(x)=f(x)$,

(ii) $f'(x)=x+f(x)$, and (iii) $f'(x)=xf(x)$, with the added condition that $f(0)=1$ in each case. Formal tricks lead to their solutions respectively as $f(x)=e^x$, $f(x)=2e^x-x-1$, and $f(x)=e^{x^2/2}$. Of course, the exponential function and its properties are taken as known. However, if we let the students on their own, successively adding little growths over sub intervals of $[0,x]$ of length x/n, they come out with their approximate solutions as $(1+\frac{x}{n})^n$, $2(1+\frac{x}{n})^n-x-1$, and $\prod_{k=1}^{n}(1+\frac{(k-1)x^2}{n^2})$. This spurs them on to investigate the sequence $(1+\frac{x}{n})^n$, (which is also important as compound interest formula). Importantly, they are led to an identity $e^{x^2/2} \approx \prod_{k=1}^{n}(1+\frac{(k-1)x^2}{n^2})$.

This last identity, the students may not possibly see in their life-time. The tentative discoveries obtained by adding differentials are, of course, legitimized by the Riemann theory of integration aided if necessary by a rigorous course in Differential equations. The plea for banishing differentials from introductory Calculus seems to argue for a total break with the historical origins and development of the subject. Is it not fair to give our students a glimpse of how tables were and are constructed? Napier's tables were constructed by adding differentials or little changes .

June 8, 1981: Miscellaneous Observations

Rigor:

For the majority of teachers and users of Calculus, a knowledge of its Foundations is no more indispensable than a knowledge of the thermodynamics of combustion engines is indispensable for those learning to drive an automobile competently. Rigor can wait for a later and more mature and leisurely hour. It is not necessary to know the embryology of eggs to make a decent and even eminently tasty omelette. Inculcation of creative and constructive habits are more important. 'Give them chastity and continence, only not yet'.

Textbooks:

Someone ought to undertake a survey of 'the most important textbooks of Calculus' published in the twentieth century, following the model of G. Bohlmann's 'Übersicht uber die wichtigsten Lehrbücher der Infinitesimalrechnung', D.M.V. vol. 6 (1899), p.92-110.

H.N. Gupta
June 1981.

3921 West 13th Avenue
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15 June 1981

Dr. R. A. Staal
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Dear Ralph:

Last week I looked up De Morgan's treatment of the natural logarithm, and found it quite different from mine. Copies of the pertinent pages of his book are enclosed.

As you can see, De Morgan's work and mine overlap only slightly. De Morgan starts out from a different place (he already has natural logarithms), and on pages 56 and 57 proves only that $f(a)$, which I called $\ell(a)$, is a constant multiple of a logarithm. His method is different from mine. Only later (page 75) does he prove that $f(a) = \ell(a) = \log_e a$.

Gupta seems to have thought that my talk was about the formula for the derivative of a^x, whereas it was really about the introduction of the natural logarithm. The essential thing was an easy proof of the fact that ℓ is a logarithm. De Morgan obtained this result only by an argument and approach unsuited to the level of instruction I had in mind. Incidentally, the derivation of the formula $f_a'(x) = f_a'(0)\, a^x$, for $f_a(x) = a^x$, is in current textbooks (e.g. Loomis, Marsden-Weinstein).

Here are some comments about the issues that you circulated, beginning with No. 1, Is Calculus the same as Introductory Analysis?

1. No! Calculus is largely concerned with calculation!

2. I think that rigor is a bad concept in elementary instruction. I suggest replacing it by honesty. Rigor is honesty on only one, technical, level. It's possible, and indeed fairly common, to be "rigorous" while being quite dishonest in a wider sense. The replacement of Calculus by Analysis, that you mention in No. 1, could be an example if the Analysis is totally unmotivated. In the jargon of the late 1960's, this doesn't "tell it like it is."

   So let me modify your question: How honest should one be, and when? Answer: To the best of our ability, we should be very honest, almost always. As far as we are able (and we shouldn't underestimate the difficul-

ties) we should tell the truth and nothing but the truth. Not always the whole truth, but as much of the whole truth as seems suitable.

3. Regarding differentials, it seems to me that there are two aspects that enter into elementary courses. One is the manipulative aspect, and the other is the "infinitesimal" or "little bit of" idea. When I refer to the first I'll use the term "differential," and use the term "infinitesimal" for the second.

The differential notation is a lot like the Hindu-Arabic notation for positive integers. It allows us to do a lot correctly, without thinking very hard about what we're doing, and allows people who don't know much theory to act correctly. We can teach, on an operational level, things that students would not be ready for on a theoretical level (though we still must be honest). This applies to pre-Analysis calculus, generally, and to differentials and infinitesimals in particular. I myself regard this as socially valuable.

Differentials and infinitesimals are certainly not necessary, but they are useful. All (correct) manipulations with differentials can be justified, so differentials aren't magic. [I do believe, though, that many students prefer magic, which is why they use integration by parts on almost anything!]

Infinitesimals are useful heuristically as well as mnemonically. A good student might, for example, anticipate the formula for the area of a surface of revolution using infinitesimals, but be baffled if he doesn't know about them. I've read something to the effect that Archimedes used infinitesimals heuristically to obtain his results, before proving them. As with differentials, all correct results obtained from a good intuition of infinitesimals can be justified in other ways. (I'm not talking about non-standard analysis.)

My conclusion is that both differentials and infinitesimals should remain in elementary calculus courses, but that we should be honest about what we're doing with them.

4. "Funny" functions can, and should, arise <u>naturally</u>. [which is not to say that we can't give nature a little push now and then.] We should have enough of them available for clarifying student misconceptions, as they arise.

5. Answer to question: The way you did it at the meeting! [But I would use x- and y-axes when using rectangular coordinates.]

6. Answer: I somehow think so, but the matter won't be settled until an acceptable compact text is written. [Lang's Calculus unfortunately cut corners.] In the meantime, note that much of a mammoth text goes unused in any one course.

7. We do need f' and dy/dx if we are not to withdraw from society. As I see it, the notation f' should be used for derivatives of number-to-number mappings, and dy/dx should be used for the rate of change of one variable quantity y with respect to a variable quantity x. I don't think we really need any other symbols for single-variable derivatives. If $y = f(x)$, where x and y are variable quantities, we shouldn't write y' except as an <u>abbreviation</u> for dy/dx. Students shouldn't get the impression that y' is a complete notation, or they will get really hung up when they do related rates.

8. Some applications do have a special status. Those associated with position-velocity-time are examples. They should be included for historical, general education, and "comprehension" reasons. As I said at the meeting, applications should be brought into mathematics courses mainly to illuminate the mathematics. [And mathematics should be brought into courses in other fields mainly to illuminate these other fields.] Some applications of calculus, such as to arc length and area, should of course be to other parts of mathematics.

9. I definitely don't believe in trivialized calculus courses. If calculus, per se, is done in high school then it should be done well, <u>without</u> a restriction fo rational functions. But this doesn't mean that calculus ideas shouldn't be brought in unless you give a complete calculus course. I believe that some thought should be given to introducing calculus ideas <u>very early</u> in high school (ninth grade, perhaps?), to be studied <u>in conjunction with</u> algebra, geometry, and trigonometry, all through high school. This would make high school mathematics courses a lot juicier.

10. <u>Basic</u> techniques should be taught as a unit. (But I'm not sure.) More advanced techniques could be taken up later, after other material is covered.

11. The way I did it at the meeting!

12. Yes.

13. As stated, a stronger role than is provided by one-paragraph biographical inserts. Some of the history should <u>connect</u> with the mathematics.

14. Yes, certainly, especially if you count the "approach" as part of the proof. For example, the Cauchy Mean Value Theorem has a simple, natural standard proof if you say the right things in the right order:

    A. Bring in the geometric meaning, in terms of slopes of secants and tangents to paths of moving particles. This yields, intuitively, $[g(b) - g(a)]/[f(b) - f(a)] = g'(c)/f'(c)$ for some c between a and b.

    B. This statement is essentially equivalent to $[g(b) - g(a)]\, f'(c) - [f(b) - f(a)]\, g'(c) = 0$ for some c.

    C. So consider the function $h(x) = [g(b) - g(a)]\, f(x) - [f(b) - f(a)]\, g(x)$ and apply Rolle's theorem, in the version that doesn't require $h(a)$ to be 0.

    D. State the theorem formally, noting the conditions to be imposed on f and g.

    I don't claim that this approach and proof are "best," for all time. But they certainly aren't "worst." A worse <u>proof</u> would involve a more complicated, less natural function. For a worse <u>approach</u>, just omit the geometric meaning and begin with the formal statement! [See Shanks and Gambill's <u>Calculus</u>, 2nd edition, page 352.]

I hope that these remarks will be of some value to you.

Yours sincerely,

Bob.

Robert R. Christian

Another way to introduce natural logarithms and e

by Robert R. Christian

The purpose of the present note is to give a simple proof of a well-known theorem, and to indicate how this theorem, and its proof, might be used as the basis of a very natural introduction to the natural logarithm function and its base e.

Here's the theorem:

Theorem. For each positive number a let

$$\ell(a) = \lim_{h \to 0} \frac{a^h - 1}{h}$$

and suppose that ℓ is a well-defined function not identically zero. [Please pardon me for packing the hypothesis.] Then ℓ is a logarithm function.

Proof. For arbitrary numbers r and positive numbers a, $\ell(a^r) = \lim_{h \to 0} (a^{rh} - 1)/h$ $= r \lim_{h \to 0} (a^{rh} - 1)/(rh) = r\,\ell(a)$. For $\ell(a) \neq 0$, let $b = a^{1/\ell(a)}$, and let $x > 0$ be arbitrary. Then $\ell(b) = [1/\ell(a)]\,\ell(a) = 1$ and $x = b^{\log_b x}$, so $\ell(x) = \ell(b^{\log_b x}) = (\log_b x)\,\ell(b) = \log_b x$.

Thus ℓ is the logarithm function $\log_b$. [This proof also shows that the only non-zero functions f for which $f(a^r) = r\,f(a)$ are the logarithm functions.]

The natural logarithm function is now defined as the function ℓ of the theorem, and its base is called e. This function arises naturally from trying to find the derivatives f_a' of the functions $f_a(x) = a^x$ via limits of difference quotients or, on a less abstract level, slopes of the curves $y = a^x$:

$$f_a'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = a^x \ell(a) .$$

To use the theorem and its proof as the basis of a very natural introduction to the natural logarithm function one might begin by assuming, explicitly, certain intuitive and computational facts about exponents and logarithms normally covered in high school. This would include the existence and differentiability of the functions $f_a(x) = a^x$, covered in a primitive way when the student is told that for each $a > 0$ the points (q, a^q), with q rational, may be connected with a <u>smooth</u> curve. Specifically, one might begin by assuming the following, or less:

For each positive number a there exists one and only one function f_a, defined and differentiable everywhere, such that $f_a(q) = a^q$ for each rational number q, $f_a(x) > 0$ for each real number x, and $f_a'(x) \neq 0$ for at least one $a > 0$ and at least one x. Furthermore, if we denote $f_a(x)$ as a^x, then

$$a^{x+y} = a^x a^y ,$$

$$a^{x-y} = a^x / a^y ,$$

and $$(a^x)^y = a^{xy}$$

for all numbers x and y. And, if $a \neq 1$ then for each $X > 0$ there is one and only one number x such that $a^x = X$. This number is denoted as $\log_a X$, and

$$\log_a(XY) = \log_a X + \log_a Y ,$$

$$\log_a(X/Y) = \log_a X - \log_a Y ,$$

and $$\log_a(X^r) = r \log_a X$$

for all numbers r and all positive numbers X and Y.

[The fact that f_2', say, is not identically zero could be regarded as intuitively clear from the graph of f_2, or, alternatively, proved easily with the help of the Mean Value Theorem. Since $f_2'(x) = 2^x \ell(2)$, this would

be sufficient to show that the function ℓ of the above theorem was not identically zero.]

From this beginning everything would go quickly and naturally. We would find the derivatives f_a' as indicated above, to obtain the formula $f_a'(x) = a^x \ell(a) = a^x \log_e a$. In this way the natural logarithm function and its base would be forced upon us with some immediacy. The number e would not be chosen at all. In particular, it would not be chosen merely to sweep a messy constant under the rug, nor would its choice require a vague intermediate value argument. And the formula $f_a'(x) = a^x \log_e a$ would be obtained whole, without recourse to the chain rule.

Of course this introduction would not be a mathematical development of the natural logarithm from scratch, but simply a piece of pedagogy. Obviously it would bypass the hard work, which would be to prove that the above "high school" assumptions are valid. But no law says that proofs may not be postponed. Indeed, one could argue that introductions shouldn't involve too much hard work, and that proofs should be given when the time is right for them. With the above introduction to the natural logarithm it

would seem to me that the "right" time for proof would be at some time after an introduction to the (definite) integral. By this time the student would have learned that $f'(x) = 1/x$ if $f(x) = \log_e x$, and that $\log_e x = \int_1^x (1/t)\,dt$. With this much to work with it would be natural to investigate the function $L(x) = \int_1^x (1/t)\,dt$ from scratch, in order to prove the validity of the "high school" assumptions made some time before. [This is, incidentally, how many textbooks introduce natural logarithms.] The student reaching this stage should have a good idea of what to expect in the way of intermediate results and technical details, and thus should be enlightened by the route taken, rather than puzzled.

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Now let $u = x^{-p}$, where p is positive, whole or fractional

$$u = \frac{1}{x^p} \text{ (p. 52)} \quad \frac{du}{dx} = \frac{x^p \frac{d1}{dx} - 1 \times \frac{dx^p}{dx}}{x^{2p}}$$

$$\frac{d1}{dx} = 0, \quad \frac{d.x^p}{dx} = p x^{p-1} \text{ (by the two last cases)}$$

$$\frac{du}{dx} = -\frac{p x^{p-1}}{x^{2p}} = (-p) \,.\, x^{p-1-2p} = (-p)\ x^{(-p)-1}, \left(\begin{matrix}\text{p. 35, Rule 2}\\ \text{in part.}\end{matrix}\right)$$

Let $u = a^x$, which gives

$$\frac{\Delta u}{\Delta x} = \frac{a^{x+\Delta x} - a^x}{\Delta x} = a^x \times \frac{a^{\Delta x} - 1}{\Delta x},$$

and the question is now reduced to finding what limit has $\frac{a^\theta - 1}{\theta}$ when θ diminishes without limit, the singular form being $(\theta = 0)\ \frac{a^0 - 1}{0}$ or $\frac{1-1}{0}$ or $\frac{0}{0}$, as in other cases. This limit must be some function of a, for θ cannot appear in a function which (when a proper form is given to it) is found by making $\theta = 0$. For the same reason, the limit of $\frac{a^x - 1}{x}$ is the same function of a, if x diminish without limit. We obtain, therefore, the same limit if x be a function of θ, provided both diminish without limit together. Let $x = b\theta$, b being a constant. Then we have

$$\text{limit } \frac{a^{b\theta} - 1}{b\theta} = \text{limit } \frac{a^\theta - 1}{\theta} \ldots\ (1.)$$

But $\frac{a^{b\theta} - 1}{b\theta} = \frac{1}{b} \cdot \frac{(a^b)^\theta - 1}{\theta}$, which second factor only differs from $\frac{a^\theta - 1}{\theta}$ in having a substituted for a, and therefore its limit is the same function of a^b, which that of $\frac{a^\theta - 1}{\theta}$ is of a. Let the limit of this latter be $f a$, then we have

$$\text{limit } \frac{1}{b} \frac{(a^b)^\theta - 1}{\theta} = \frac{1}{b} \text{ limit } \frac{(a^b)^\theta - 1}{\theta} = \frac{1}{b} f(a^b),$$

consequently (1) the function $f a$ is such that

$$\frac{1}{b} f(a^b) = f(a) \text{ or } f(a^b) = b f a,$$

and a and b are independent of each other. If a^b be q, we have $\log_a q = b \log a$, whatever the base of the logarithms may be. This gives

$$f(q) = \frac{\log q}{\log a} f a \text{ or } \frac{fq}{\log q} = \frac{fa}{\log a},$$

and q and a may have any different values we please, for though $q = a$, yet since b may be what we please, it may be so taken (exactly or with any degree of approximation we please) as to give q any other value.

Therefore $f a$ is such a function as to give $\frac{fa}{\log a}$ this property, that it remains the same if any other quantity q be substituted for a. That is,

$$\frac{fa}{\log a} \text{ is a constant independent of } a, \text{ which call C.}$$

$\therefore fa = \text{C} \log a$; but the equation

$$\frac{\Delta u}{\Delta x} = a^x \frac{a^{\Delta x} - 1}{\Delta x} \text{ gives } \frac{du}{dx} = a^x \times \text{limit } \frac{a^{\Delta x} - 1}{\Delta x}$$

or $\frac{du}{dx} = \text{C} \log a \times a^x$, where all that is known of C is, that it is independent of a. It must clearly depend on the *base* of the logarithms chosen, and it will afterwards be shown that *when the logarithms are Naperian, then* C = 1. But this point must be reserved till the next chapter. Remember, that for the present, all differentiations which contain a^x are not finally demonstrated until it shall have been shown that if $u = a^x$, $\frac{du}{dx} = \text{Nap. log } a \times a^x$; all we know is that, taking these logarithms, it must be of the form C Nap. $\log a \times a^x$ where C is not determined, but assumed, for the present, to be $= 1$.

From this it will follow that if $a = e = 2{\cdot}7182818 \ldots$. the base of Napier's logarithms, or if $\log e = 1$, and if $u = e^x$, $\frac{du}{dx} = 1 \times e^x = e^x$, (p. 36, Rule 4.)

Let $u = \log x$ to the base a or $x = a^u$

then $\frac{dx}{du} = a^u \times \log a = x \log a$

$$\frac{du}{dx} = \frac{1}{\frac{dx}{du}} = \frac{1}{x \log a} = \frac{\text{M}}{x},$$

where M is the modulus * of the system of logarithms having a for its base. Hence, since $\log a = 1$,

$$\text{if } u = \log x \quad \frac{du}{dx} = \frac{1}{x} \text{ (p. 36, Rule 3.)}$$

(Read here the proof that the limit of the ratio of $\frac{\sin}{\theta}$ is 1 when θ diminishes without limit, given in the 'Elementary Illustrations,' &c. p. 4.)

Let $u = \sin x$, $\Delta x = \theta$, $\frac{\Delta u}{\Delta x} = \frac{\sin(x+\theta) - \sin x}{\theta} = \frac{2\cos\left(x + \frac{\theta}{2}\right) \sin \frac{\theta}{2}}{\theta}$

$$= \cos\left(x + \frac{\theta}{2}\right) \times \frac{\sin \frac{\theta}{2}}{\frac{\theta}{2}}, \text{ whose limit is } \cos x \times 1.$$

* By a well-known relation, $\log a$ (to base y) $\times \log y$ (to base x) $= 1$.
Hence $\frac{1}{\log a \text{ (base } x)} = \log x \text{ (base } a) =$ Modulus of system whose base is a.

Paper #9

where, however, it must be observed, that though θ is less than unity in every one of these cases, it is not the same in all.

$$\sin(a+h) = \sin a + \cos(a+\theta h)\,.\,h$$

$$= \sin a + \cos a\,.\,h - \sin(a+\theta h)\,\frac{h^2}{2}$$

$$= \sin a + \cos a\,.\,h - \sin a\,\frac{h^2}{2} - \cos(a+\theta h)\,\frac{h^3}{2.3}, \text{ \&c.}$$

We shall ascertain the truth of the first line by an instance, which will also serve to illustrate the way in which angles are measured in analysis (a point on which the notions of most students are remarkably confused: see Penny Cyclopædia, article Angle, 'Study of Mathematics,' p. 89.) Let a be (in common degrees and minutes) 35°, and let h be 10°. When these enter under a sine or cosine, it is most convenient to express them in degrees, minutes, &c., because the sines, &c. are given to those denominations in the tables, and are the same for the same angles in whatever way we may measure the angles. But when an angle enters as an angle, the truth of all theorems yet obtained depends upon measuring that angle by the fraction which its arc is of the radius*.

The angle of 10° must be expressed by ·1745329. The assertion then which we wish to verify amounts to this: that if we find θ from the equation

$$\sin(35° + 10°) = \sin 35° + \cos(35° + \theta \times 10°) \times \cdot 1745329$$

we shall find it less than unity.

$\sin 45° = \cdot 7071068$	$\log \cdot 1335304$	$\bar{1}\cdot 1255801$
$\sin 35° = \underline{\cdot 5735764}$	$\log \cdot 1745329$	$\bar{1}\cdot 2418773$
$\cdot 1335304$	$\log \cos. 40° 5'$	$\bar{1}\cdot 8837028$

$$35° + \theta \times 10° = 40° \tfrac{1}{10}' \qquad \theta = \frac{5\frac{1}{10}}{10} = \cdot 501 = \tfrac{1}{2} \text{ nearly.}$$

We now come to a modification of the preceding, which is usually called Maclaurin's Theorem, but which should be called Stirling's Theorem†. If we suppose $a = 0$ to satisfy the conditions under which Taylor's Theorem exists, that is, if we suppose $f0, f'0, f''0 \ldots$ to be all finite up to $f^n 0$ we have, by Taylor's Theorem,

$$f(0+h) = f0 + f'0\,.\,h + f''0\,\frac{h^2}{2} + f'''0\,\frac{h^3}{2.3} + \ldots + f^n 0\,\frac{h^n}{2.3\ldots n} + f^{(n+1)}(0+\theta h)\,\frac{h^{n+1}}{2.3\ldots(n+1)},$$

and remembering that h being anything whatever, we may write x for h, we have

* It may be worth while to revert to the fundamental step on which this rests. It is a theorem derivable from 'Elementary Illustrations,' p. 5., that the limiting ratio of a comminuent sine and angle is 1. Now this theorem is not true of *the number of seconds in an angle*: but only of the fraction which the arc of the angle is of its radius.

† Maclaurin, in our view of the subject, was the first who wrote a logical treatise on Fluxions. The reader who would verify the assertion implied in the text for himself must compare Stirling's 'Methodus Differentialis,' London, 1730, p. 102, "Hinc si ordinata Curvæ, &c." with Maclaurin's Fluxions, Edinburgh, 1742, p. 610, "The following theorem, &c." The fact, we doubt not, would be, that both Maclaurin and Stirling would have been astonished to know that a particular case of Taylor's theorem would be called by either of their names.

$$fx = f0 + f'0\,.\,x + f''0\,\frac{x^2}{2} + \ldots + f^{(n)}0\,\frac{x^n}{2.3\ldots n} + f^{(n+1)}\theta x\,\frac{x^{n+1}}{2.3\ldots n+1},$$

of which the following is an instance:—

$fx = \sin x$, $f'x = \cos x$, $f''x = -\sin x$, $f'''x = -\cos x$, $f^{iv}x = \sin x$, &c.

$f0 = 0$ $\quad f'0 = 1$ $\quad f''0 = 0$ $\quad f'''0 = -1$ $\quad f^{iv}0 = 0$, &c.

$$\sin x = 0 + \cos\theta x\,.\,x = 0 + 1 \times x - \sin\theta x\,\frac{x^2}{2}$$

$$= 0 + 1 \times x - 0 \times \frac{x^2}{2} - \cos\theta x\,\frac{x^3}{2.3}$$

$$= 0 + 1 \times x - 0 \times \frac{x^2}{2} - 1 \times \frac{x^3}{2.3} + \sin\theta x\,\frac{x^4}{2.3.4}, \text{ \&c.}$$

or $\sin x = \cos\theta x\,.\,x = x - \sin\theta x\,\frac{x^2}{2} = x - \cos\theta x\,\frac{x^3}{2.3}$

$$= x - \frac{x^3}{2.3} + \sin\theta x\,\frac{x^4}{2.3.4} = x - \frac{x^3}{2.3} + \cos\theta x\,\frac{x^5}{2.3.4.5}$$

$$= x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \sin\theta x\,\frac{x^7}{2.3.4.5.6.7}, \text{ \&c.;}$$

where θ is not (as far as we know) the same fraction in any two, but in all is less than unity. The first one is a remarkable relation, and may be expressed thus: a sine divided by its angle is the cosine of a smaller angle.

We now proceed to the completion of the process of differentiation, by determining the value of the constant which enters $\frac{du}{dx}$ where $u = a^x$, having found that if $\phi x = a^x$ $\phi' x = C \log a\; a^x$, $\phi'' x = (C \log a)^2 a^x$, &c.

This gives, by Taylor's theorem,

$$a^{x+h} = a^x + C \log a\,.\,a^x\,.\,h + (C \log a)^2 a^x\,\frac{h^2}{2} + \ldots + (C \log a)^n a^x\,\frac{h^n}{2.3..n} + (C \log a)^{n+1} a^{x+\theta h}\,\frac{h^{n+1}}{2.3..n+1} \qquad \theta < 1.$$

Now the value of C depends upon the base of the logarithms chosen; which base being generally derived from an infinite series, we shall not take it for granted, but reverse the question: that is, instead of asking what must C be when the base chosen is 2·71828 .... usually called e, we shall ask, what must that base be for which C is 1. Or given $C = 1$ to determine a. Taking the value of a for the base, we have $\log a = 1$, and taking $C = 1$, we have to determine a from this equation (derived from the preceding by dividing by the common factor a^x, and substituting 1 for $\log a$ and for C)

$$a^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{2.3} + \ldots + \frac{h^n}{2.3..n} + a^{\theta h}\,\frac{h^{n+1}}{2.3\ldots n+1}.$$

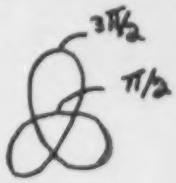
This will be true, for the proper value of a, whatever h may be: let us therefore make $h = 1$, which gives

$$a = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \ldots + \frac{1}{2.3..n} + \frac{a^\theta}{2.3..n+1} \qquad \theta < 1;$$

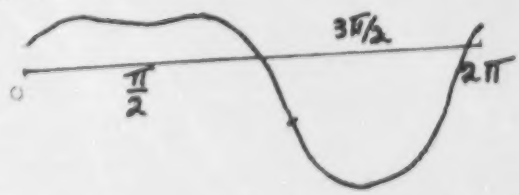
I've found rather a nice example where your suggestion of sketching a graph of r against θ (as the first step in sketching a polar graph) works better than the usual technique of making a table of relevant values of r and θ.

The graph is $r = 1 + 2\sin\theta - 2\sin^2\theta$.

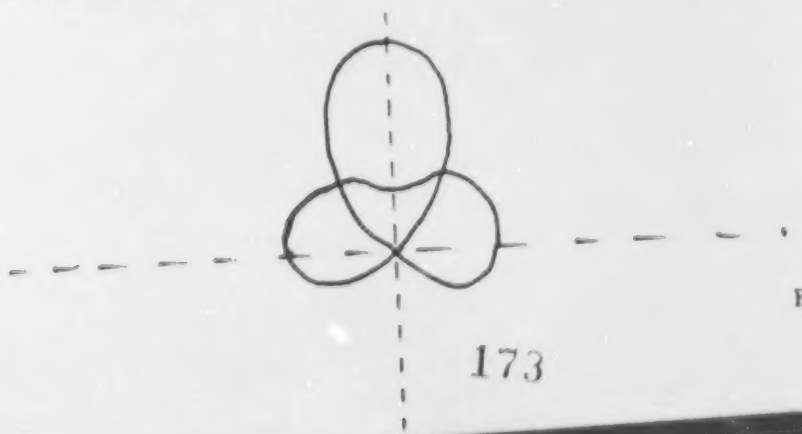
From a table most people would sketch it as



but the graph of r against θ looks like this



and suggests to a cautious person that the graph might be concave up at $\theta = \frac{1}{2}\pi$. If we check stationary values for y $(= r.\sin\theta)$ we find quite easily that it is, and the correct graph is



H.Thurston

Paper # 11

UNE APPROCHE FONDAMENTALISTE DE L'ENSEIGNEMENT DU CALCUL INFINITESIMAL

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A- INTRODUCTION

L'objet de cet exposé est de donner la problématique d'un projet de recherche présentement en cours à l'Université du Québec à Rimouski. Plus précisément, nous donnerons:

1- le contexte et la dynamique du développement de ce projet de recherche;

2- une ébauche de ce projet de recherche.

B- CONTEXTE SCOLAIRE QUEBECOIS

Le système scolaire québécois est structuré de la façon suivante: la maternelle, l'école élémentaire d'une durée de six ans, l'école secondaire d'une durée de cinq ans, les Collèges d'enseignement général et professionnel (C.E.G.E.P.) d'une durée de deux ans pour le secteur général (Sciences Pures, Sciences de la Santé, etc.) et d'une durée de trois ans pour le secteur professionnel (Sciences infirmières, techniques, etc.) et ensuite l'Université d'une durée de trois ans pour l'obtention d'un baccalauréat.

C- SITUATION DE L'ENSEIGNEMENT DES MATHEMATIQUES AU NIVEAU COLLEGIAL

Le ministère de l'Education a fixé comme objectif premier à l'enseignement des mathématiques au collégial, de mathématiser des situations concrètes. Or point n'est besoin d'une recherche exhaustive pour se rendre

compte que cet objectif ne s'est jamais concrétisé, en général.

Un des facteurs qui a contribué à cet état de fait est sans conteste le transfert du niveau universitaire vers le niveau collégial, de la conception de l'enseignement des mathématiques et du type de mathématiques à enseigner, et ce, surtout lors de la création des collèges. Ceci a eu pour effet d'expulser de l'enseignement des mathématiques au collégial toute application de la vie courante, et de formaliser à outrance ce même enseignement. En même temps que ceci se produisait, l'Etat mettait l'accent sur la démocratisation de l'éducation. Il n'en fallait pas plus pour obtenir quelques générations de "drop-out" des mathématiques et provoquer le mouvement de balancier vers l'autre extrême qui consiste à introduire dans l'enseignement des mathématiques un pragmatisme étroit. Dans un premier temps nous avons donc perdu de vue le sens éminemment pratique des mathématiques, pour ensuite se détourner de sa signification originelle. Il n'est pas étonnant, dans un tel contexte, que les étudiants ne comprennent pas le rôle que les mathématiques jouent dans notre société. Nous sommes donc à la recherche d'un juste équilibre entre l'aspect utilitaire des mathématiques et son aspect purement formel.

D- MAITRISE EN MATHEMATIQUES ORIENTATION ENSEIGNEMENT

L'Opération Sciences Fondamentales dans son rapport au ministère de l'Education du Québec pp. 234-235 signale le manque d'adéquation entre les programmes de formation actuels pour les enseignants des niveaux

pré-universitaires et les fonctions de ces mêmes enseignants. C'est, en particulier, pour éliminer cette lacune que l'Université du Québec a mis sur pied une maîtrise en mathématiques orientation enseignement offerte depuis janvier 1979 dans ses constituantes de Chicoutimi, Rimouski et Trois-Rivières. Ce programme vise essentiellement à parfaire la formation des enseignants de mathématiques, notamment dans le domaine de la pédagogie de la mathématique. On doit comprendre que l'acquisition de connaissances nouvelles en mathématiques ne constitue pas la première priorité de ce programme bien qu'elle y soit présente. La majorité des activités de ce programme est reliée directement à la méthodologie de l'enseignement des mathématiques. Ces activités représentent à la fois l'objectif principal et le fil conducteur du programme: on se propose en effet de reprendre les notions mathématiques déjà connues, de les replacer dans l'histoire, d'établir les liens qui existent entre elles, d'étudier les différentes méthodes d'enseignement de ces notions et enfin, de faire l'étude des nombreuses applications possibles au niveau collégial dans le contexte de l'étude du processus de mathématisation. La structure générale de ce programme est la suivante: 12 crédits de cours de didactique des mathématiques, 18 crédits de cours de mathématiques ou d'histoire des mathématiques, 3 crédits de séminaire et 12 crédits de recherche.

E- COURS DE DIDACTIQUE DE L'ANALYSE

Certains cours de la maîtrise sont offerts simultanément dans toutes les constituantes à l'aide du système vidéo, système de télévision

en circuit fermé bidirectionnel au niveau de l'image avec un centre et bidirectionnel au niveau du son avec tous les centres.

Un des cours offert à l'hiver 1981 à l'aide du système vidéo fût le cours de Didactique de l'analyse. L'objectif fondamental de ce cours était de mieux préparer l'étudiant-maître à enseigner les cours de mathématiques du collégial reliés à l'analyse et au calcul infinitésimal. A cet effet les activités ont été choisies en vue:

i) de susciter une réflexion d'ordre didactique sur les concepts de base en analyse;

ii) de favoriser chez l'étudiant-maître une réflexion sur les problèmes liés à l'enseignement du calcul infinitésimal;

iii) d'étudier et d'illustrer le processus de mathématisation de situations concrètes.

Dans une perspective de leur enseignement, les concepts de base de l'analyse ont été abordés d'un point de vue mathématique et historique. Du point de vue mathématique, l'étude visait à faire ressortir la pertinence des définitions ainsi que l'importance et la nécessité des théorèmes. Ceci a permis à l'étudiant-maître d'une part, de mieux comprendre les subtilités internes de la mécanique de cette matière et d'autre part, de mieux juger de l'importance des notions qu'il a à enseigner. Du point de vue historique, nous avons dégagé les principales raisons qui ont donné naissance aux concepts du calcul infinitésimal et de l'analyse, et nous nous sommes attardés à percevoir l'influence de l'histoire sur leur enseignement.

Pour ce qui est de la réflexion d'ordre pédagogique, elle visait à susciter une interrogation à la fois sur la pertinence de l'enseignement formel de certains concepts, dit de base, dans l'enseignement du calcul infinitésimal et sur la compréhension minimale que l'on doit exiger de ces concepts chez les étudiants du collégial. De plus, l'étudiant-maître a été amené à analyser différentes présentations de certains de ces concepts.

Un des grands courants de pensée qui a marqué l'humanité est, sans conteste, la mathématisation de situations concrètes. Pour le réaliser on n'a qu'à penser aux réussites qu'a connues et que connaît encore la méthode scientifique. Comme le calcul infinitésimal a été, jusqu'à récemment, presque l'unique outil mathématique de développement, utilisé par ce processus, il était tout naturel de lui réserver une place de choix dans un cours de Didactique de l'analyse. Il y a donc eu des activités où l'on s'interrogeait sur la philosophie sous-jacente au processus de mathématiser, ainsi que sur le processus lui-même de mathématisation. De plus, nous avons illustré notre propos, en présentant plusieurs applications du calcul infinitésimal pouvant être transférées aux étudiants du collégial.

Tout au long des activités concernant les objectifs précédents l'étudiant-maître a été mis en présence de techniques propres à l'analyse et au calcul infinitésimal et souvent il a été amené à s'interroger sur le rôle de ces techniques dans son enseignement.

Le cours fût constitué de cinq (5) parties. Ce sont:

I- Introduction

- Le calcul infinitésimal et l'analyse: leurs histoires, rôles et influences.

II- Limites de suites et fonctions

- Présentation du concept de limite de suite à des étudiants du collégial, en utilisant la méthode "par découverte".
- Une présentation formelle des concepts de limite de suites et de fonctions et des principaux théorèmes.
- Atelier sur l'enseignement des concepts de limite.

III- La dérivée et l'antidérivée

- Une présentation du concept de dérivée et des principaux théorèmes.
- Présentation d'un modèle du processus de mathématisation et présentation d'une application.
- Atelier sur l'enseignement de la dérivée.

IV- L'intégrale et le lien qu'elle a avec la dérivée

- Une présentation du concept d'intégrale et des principaux théorèmes.
- Les techniques d'intégration.
- Présentation d'applications.
- Atelier sur l'enseignement de l'intégrale.

V- Renouvellement du curriculum et de l'enseignement du calcul infinitésimal

- Les méthodes numériques d'intégration.
- Les séries et les séries de puissance: un autre point de vue.
- Différentes approches de l'enseignement du calcul infinitésimal.
- Atelier sur l'enseignement du calcul infinitésimal: les objectifs, les prérequis, le sens de la rigueur, le rôle de l'histoire, le rôle des applications, l'évaluation.

F- EBAUCHE DU PROJET DE RECHERCHE

Lors de la préparation du cours de Didactique de l'analyse, nous pressentions la nécessité d'introduire au syllabus précédent un chapitre sur les processus d'apprentissage du calcul différentiel et intégral. Cependant la carence de recherche sur ce sujet nous obligea a abandonner cette idée. Cependant elle nous fit voir la possibilité de démarrer une recherche originale sur ce même sujet. Nous avons donc élaboré un projet de recherche intitulé "Modèle didactique de l'apprentissage du calcul infinitésimal, pour le niveau collégial".

Nous avons divisé la réalisation de ce projet de recherche en trois étapes:

I- Conceptualisation du modèle

La première étape consiste à dégager à partir des structures du calcul infinitésimal et de son développement historique, des théories de

l'apprentissage et des processus cognitifs, les principes de base en vue d'élaborer le modèle. Il s'agit de déterminer, pour un thème donné du calcul infinitésimal, le corpus de la théorie, les étapes d'apprentissage, les modes de compréhension et les modes de représentation les plus appropriés pour ce thème. Ces thèmes sont les fonctions, les limites, la continuité, la dérivabilité et l'intégrabilité.

II- Concrétisation du modèle

La deuxième étape consiste d'abord à déterminer, pour ce même thème du calcul infinitésimal, des situations présentant des stimuli présumément favorables à l'apprentissage. Ces stimuli peuvent être, soit des analogies ou des métaphores aidant à faire comprendre une notion du calcul infinitésimal, soit une application rendant signifiante une telle notion, soit l'utilisation d'un ordinateur ou d'une calculatrice de poche pour simuler certains processus et leur donner ainsi un sens concret, soit la donnée de problèmes qui mettent en relief la théorie ou qui exploitent le caractère ludique de l'étudiant en lui permettant de relever un défi intellectuel. Ces situations seront ensuite réalisées et conduiront naturellement à des études de cas.

III- Ajustement du modèle

A la lumière des analyses des études de cas, le modèle sera réajusté et réintroduit à la deuxième étape de la réalisation du projet.

G- CONCLUSION

Le projet de recherche, dont nous vous avons présenté l'ébauche, en est présentement au stade de la conceptualisation. Cependant, lorsque le modèle sera élaboré pour au moins un des thèmes du calcul infinitésimal, il permettra de développer des stratégies d'enseignement qui intégreront mieux les contenus relatifs à ce thème du calcul infinitésimal et les aspects psychopédagogiques relatifs à ce thème dans le programme du collégial, palliant ainsi une carence évidente de l'enseignement du calcul infinitésimal.

APPENDIX D

CONTRIBUTIONS FROM

WORKING GROUP D

LIST OF PARTICIPANTS, GROUP D

H. Allen	Faculty of Education, Queen's University
T. Bates	Faculty of Education, University of British Columbia
R. Caron	conseiller pédegogique, C.S. de l'Argile Bleue
K. Harburn	Faculty of Education, University of Toronto
M. Hoffman	Brooklyn, N.Y.
A. Powell	Brooklyn, N.Y.
L. Thérien	Department of Mathematics, University of Sherbrooke
T. Thompson	Department of Mathematics, Dalhousie University

"Report" for the group D

Loïc Thérien
University of Sherbrooke
Dept. of Math. and Info.

CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

To: Hugh Allen

Here are some "concrete" situations I have used in my class. These are quite different not so much by their form, but by the use I made of them, in other words, by the purposes I had in mind.

I could try to classify them by the following criteria:

- for prospective or in-service teachers — for pupils of the elementary level
- directed towards a specific body of notions — without such a specific body of notions
- used as "starters" — used as a way of summarizing or synthetising
- used to illustrate a learning process — or used to apply notions
- very open situations (no specific questions or answers) — semi-open situations (specific questions; more than one answer)

	teachers	specific body of notions	starters	open	semi-open	illustrate a learning process	team
1. animals	X	X	X	X		X	X
2. humans	X	X		X			
3. dice and squares	X		X		X	X	X?
4. towers of equal height		X			X		X?
5. real objects to describe by geometrical notions	X	X			X		
6. $S_{2dice} = 7$	X	X	X		X	X	X

All of them could be seen as "problems". The criteria are, in no way, exclusive; they attempt to describe the main purpose or the main feature of these activities.

2, 5, and possibly 4, are done by individuals;
1, 3 and 6 are done by teams of 2, 3 or 4.

1. ANIMALS

The students are asked to list about 15 names of animals. Then, they have to find relations between them, their differences, their resemblances. Finally, they are asked to represent on a large sheet of paper what they have found, so that somebody who has not followed their previous work could vizualise easily what they have in mind. Questions are as imprecise as possible.

When they have finished, the sheets of paper are exposed all arpund the class and discussed.

First, they are explained to others.
I try with them to find the good aspects and the weaknesses.
They always have good points.
After that, we try to classify them, either by their external aspect or by commun ideas.
Usually, each class, if refined, conducts to a standard representation: Venn, Carroll, tree, a mixture, figurative, arrow-graph, cartesian graph, and so on. We look for how to refine them and we compare them by their qualities and limitations.
We try to find what are the qualities of a good representation in relation with the aims one has.
Later, we analyse the criteria they used and try to find the qualities they shoud have.

This study drives us to develop the idea of a model, especially concretisation materials like logic blocks, Cuisenaire rods,

multibase blocks, and so on. I speak about the Linné's and Darwin's models to show that the chosen criteria determine the results and notions developped later on. Then, the choice of a concrete material (that means: the choice of criteria) is determined by the notions we want to developp or study, without forgetting its limitations.
What is a good concrete material? How to use it? What are its aims and its dangers? These are questions that naturally arise.

Parallel to this study, is a "psycho-pedago-didactic" analysis: what is an open situation, what is a learning process, what is mathematics(!?), what are the stages of the learning process, what is a motivating activity for children, why starting from manipulation, what is a genetic approach, and so on.

Finally, we develop a specific vocabulary (logic and set) directly from this situation.

2. HUMANS

This activity follows the one with animals; the same questions are asked. It has to be done at home.

3. DICE AND SQUARES (Game for two)

Given this table ,

children or prospective teachers are asked to toss four dice and to inscribe the results in the 4 cases, as they wish. Then, they sum each row, each column and the main diagonal (↘). Finally, they sum up all the results. The one who has the biggest sum wins. They have to find a winning strategy.

A variation is to toss one die at a time and to place the result immediatly.

Many questions may follow:

What is the minimum sum?
What is the maximum sum?
What the numbers that are not reached?
Why?
What are the chance to reach such and such number?
In how many ways, 37 could be reached?

Variations

a) Sum also the other diagonal. Same questions.

b) Sum only rows and columns. Same questions.

c) Take the product of the sums. Same questions.

d) Take the sum of the products. Same questions.

e) One dice determine the sign: each case can be filled by x: $-6 \leq x \leq 6$. Same questions.

f) Go from 1 to 10 or from -10 to 10. Same questions.

g) Go from -n to n. Same questions.

h) Take a 3 x 3 table. Same questions.

i) Change the disposition of the squares and so on.

Body of knowledge: Probabilities, additions, multiplications, multiples, mental calculus, diophantines equations, negative numbers, and so on.

For young children, it is an interesting way of work out exercices on operations.

4. TOWERS OF EQUAL HEIGHT

Pupils are given a number of cubes (say 24) and they are asked to make towers of equal heights. These are buildings of a section of the town 24, say the section 4 (6 building of 4 levels).

At the first stage, they are asked to build all the possible sections (to construct to whole city). Then, they build many cities. Some patterns or regularities appear.

At the second stage, they have to find ways of building sections, starting from a previous section (in mathematical words, for example, doubling the height, dividing by two, tripling, and so on). They name and describe their actions.

They are invited to compose and to inverse these actions, to go from one specific section to another, from 1 to x in the city x with the shortest path, to find all the paths from y to z , and so on.

Then they construct models (straws of different colors and balls). They classify them.

Then, they draw lattices of the divisors of many numbers
A mathematical vocabulary is developped.
An "ad hoc" scenario motivates these activities.
Specific body of notions: divisors, parity, multiples, prime (elementary number theory) numbers, fondamental theorem of the arithmetic, least commun multiple, greatest commun divisor, commutativity of x, distributivity, a/b and b/c $\Longrightarrow$ a/c and so on.

It proceeds from a previous research in learning processes based on the Piaget's notion of grouping. (Thérien, Héraud, Fortier).

5. GEOMETRICAL NOTIONS

At the beginning of a course in geometry to prospective teachers (classification of solides and plane figures, polyedras, polygons, measure (lenghts, areas, volume)), students are asked to produce, at the end of the course, a work where they will draw 10 to 15 real objects, chosen in their environment, caracterize and classify them with the notions seen during the course, and approximate their area and volume by doing mental operations (like cutting, rearranging the parts, deforming, developping, and so on) in such a way that they produce standard forms (cube, prism, cylinder, sphere, and so on). They don't have to compute.

6. SUM OF TWO DICE

At the beginning of a course in Probability ans Statistics for prospective teachers, I ask the students to try to predict and to find how often they produce 7 as the sum of two dice. They have to compile their results in the way they want, to classify or organize them, to represent them in a "pleasing manner", to compare their results with their predictions and finally to explain the results (mathematical or figurative proof). This situation is a starter to the fundamental notions that will be developped later on: representations, hazard, probability, statistics, ...).

Loïc Thérien
University of Sherbrooke

DU NOMBRE PAIR A LA «PREUVE PAR 9»

Si l'on veut «revitaliser» les cours de didactique de la mathématique, il nous faudra faire vivre aux enseignants des situations qui démystifient cette discipline tout en lui gardant son caractère fascinant.

Une des séquences d'enseignement qui me semble avoir atteint particulièrement ces objectifs en est une pendant laquelle les enseignants ont été appelés à réutiliser et réorganiser plusieurs connaissances acquises séparément et emmaganisées pêle-mêle dans leur mémoire. Ces connaissances prenaient un sens nouveau, les unes expliquant les autres, alors que les nombres et les opérations devenaient plus intelligibles.

Tout commence par quelques questions simples sur les nombres pairs:

«Qu'est-ce qu'un nombre pair?»

«Comment peut-on reconnaître instantanément un nombre pair?»

Dans notre système de numération, les nombres pairs présentent des régularités qui sont dues à la base (dix)du système qui est aussi un nombre pair.

Si on multiplie 10 par 10, on obtiendra un autre nombre pair et il en sera de même avec 100 x 10, 1 000 x 10, etc ... Ce sera aussi la même chose avec 2 x 100, 4 x 100, 5 x 100, etc ...

Les deux tableaux suivants résument les observations des enseignants au cours de ces discussions

X	P	I
P	P	I
I	I	P

X	P	I
P	P	P
I	P	I

Mais la plupart des enseignants ne se sont jamais livrés à ce genre de réflexion sur les nombres pairs et ils en ont encore moins discuté avec leurs élèves. Dans la plupart sinon la totalité des cas, on a procédé par le raccourci suivant: un nombre pair est un nombre dont le dernier chiffre est 0, 2, 4, 6 ou 8. Pour les amener à se rendre compte que cet état de fait tient aux propriétés de notre système numérique, je leur demande alors: «Si vous aviez un nombre écrit en base 5, comment reconnaîtriez-vous qu'il s'agit d'un nombre pair ou impair? Le nombre $431_{(5)}$ est-il pair ou impair?»

Pour répondre à cette question, ils doivent utiliser une suite d'opérations que plusieurs d'entre eux ont déjà utilisée.

- Ils commencent par dire que 4 x 25 est un nombre pair, donc, comme on n'aura pas de reste, on n'a pas à s'occuper de ce chiffre.

- Le résultat de 3 x 5 sera cependant impair, on aura donc un reste de 1 après la division par 2. Nous allons retenir uniquement ce reste de 1 pour l'ajouter au 1 des unités, ce qui nous donnera 2, un autre pair.

- Nous pouvons alors conclure que $431_{(5)}$ est un nombre pair, malgré les apparences.

Quand ils ont utilisé deux ou trois fois cette suite d'opérations, je leur demande: «Cette suite d'opérations ne vous rappelle-t-elle pas quelques chose?»

Quelques-uns d'entre eux reconnaissent dans cette suite d'opérations les étapes de la vérification de la divisibilité des nombres par 9. Je les invite alors à l'expliquer à leurs collègues.

Ceci permet alors à certains de comprendre pourquoi cette méthode fonctionne en associant les étapes de la vérification de la parité de $431_{(5)}$ avec celle de la vérification de la divisibilité par neuf de $431_{(10)}$.

Dans le premier cas, je ne retenais pas le 4 parce qu'il représentait un nombre divisible par deux mais dans le deuxième cas, je vais retenir le 4 parce qu'il représente le reste de 4 x 100: 9. Il en sera de même du 3 et du 1 qui une fois additionnés à 4 me donneront une somme de 8 qui me permettra de conclure que 431 n'est pas un multiple de 9 mais qu'il a un reste de 8 lorsque divisé par 9.

Nous discutons alors pourquoi il est si facile de vérifier la divisibilité d'un nombre par 2 dans notre système, pourquoi il en est de même avec 5. L'examen des autres cas de divisibilité les amène à voir comment tout cela obéit à quelques règles simples.

Nous oublions ensuite tout ceci pour quelques semaines. C'est quand nous travaillons les opérations sur les nombres entiers et les méthodes de vérification des résultats que nous réexaminons le même problème d'un différent point de vue.

La plupart des enseignants se rappellent avoir appris et appliqué la méthode de la «preuve par 9» pour vérifier le résultat de leur multiplication et de leur division. Il faut dire que j'ai conservé un intérêt particulier pour cette méthode parce qu'étant enfant, je n'avais pas réussi à l'apprendre. La dernière année que j'ai enseigné, elle était présentée dans le livre que les étudiants utilisaient, c'est alors que je l'ai comprise et ai tenté de démontrer à mes élèves pourquoi çà fonctionnait. J'espère avoir réussi sinon à leur faire comprendre, au moins avoir développé chez eux la conviction qu'il y avait une explication à cette méthode.

J'ai aussi découvert que les enseignants l'appliquaient et l'enseignaient sans la comprendre, c'est pourquoi, il m'a semblé important de m'y attarder. Après quelques échanges sur les différentes manières de l'appliquer, je leur demande: «Mais pourquoi cette méthode fonctionne-t-elle?»

Personne ne semble en avoir une idée jusqu'au moment où je leur demande: «Vous vous rappelez la table de la multiplication des nombres pairs et impairs? Voyez-vous dans cette table une explication à la méthode de la «preuve par 9»?».

Certains d'entre eux commencent déjà à en avoir une bonne idée et quand on a construit la table de multiplication des classes du modulo 3:

X	0	1	2	
0	0	0	0	
1	0	1	2	
2	0	2	1	

il n'y a plus de doute chez eux. Ils savent bien que les mêmes règles s'appliqueront en modulo 9, que ce sera juste un peu plus long de construire la table.

On peut cependant prendre quelques cas au hasard, un nombre de la classe 2,(9n + 2) multiplié par un nombre de la classe 5, (9n + 5) donnera un nombre de la classe 1 parce que:

$$(9n + 2)(9n + 5) = (9n)^2 + 2(9n) + 5(9n) + 10$$

Les premiers termes de l'addition sont des multiples de 9, il n'y a donc qu'à considérer 10 et le reste de 1.

C'est souvent à ce moment aussi que certains découvrent que cette méthode n'a rien d'une preuve même si elle vous donne un bon indice de la véracité de vos résultats. Le cas où quelqu'un a interverti deux chiffres dans le résultat ne serait pas décelé par exemple. Ils découvrent aussi que d'autres règles plus personnelles peuvent s'expliquer que ces règles utiles dans certains cas sont inutiles dans d'autres et que tout cela repose au fond sur une connaissance des propriétés des nombres et des opérations.

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C.S. de l'Argile Bleue

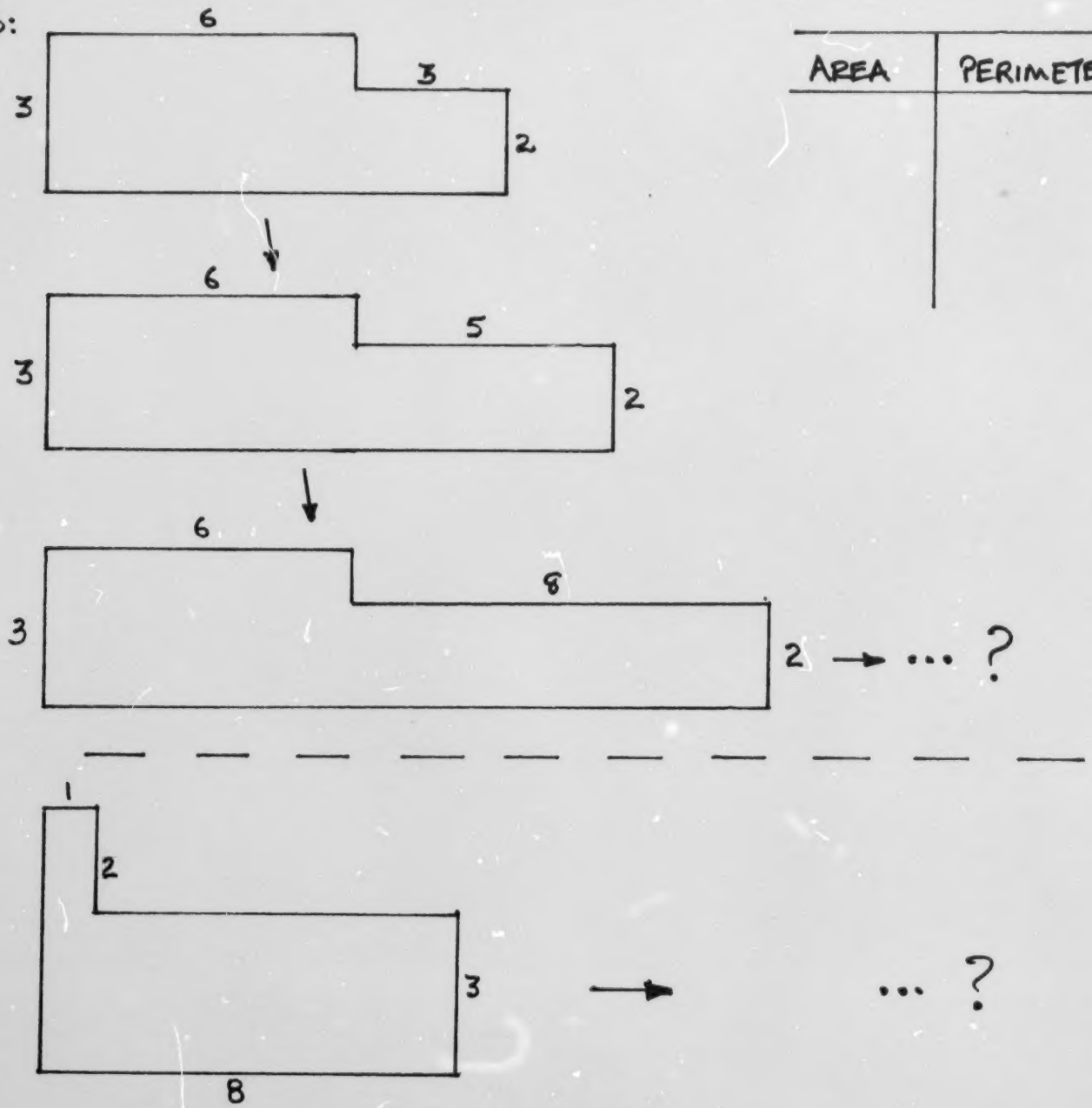
A: (Very open!)

(Presented like this, on a geoboard, or on geopaper, or on a sqare lattice, or even as an array of solid cubical blocks, nothing is suggested explicitly unless, after a reasonable interval, a student has not been able to start <u>any</u> investigation. At that point one might say, "Area? Perimeter?")

?

Alternatively, a more structured <u>sequence</u> such as this might be presented:

B:



AREA	PERIMETER

The Billiard Table

With thanks to "Mathematics A Human Endeavor", Harold Jacobs.

Our students in the B.Ed. program come to us from a variety of backgrounds but they all have had one common experience: Mathematics as an individual pursuit for right answers. Throughout their year with us, we attempt to challenge them with open-ended problems which must be investigated as a member of a group. One such problem is the "Billiard Ball Problem".

> Draw eight billiard tables with the lengths 10 and widths from 1 to 8. Show the path of a billiard ball starting from the lower left-hand corner travelling at an angle of 45° with each side of the table and mark the corner where the ball ends up each time.
>
> Change the length to 7 units and see if the rule you have developed still holds.
>
> Is it possible to predict where the ball ends up on a "n" by "m" table?
>
> On most of the tables the ball travels over every square. Can you think of a rule for telling in advance which tables these are without drawing them?

A second problem based on the billiard table also involves a piece of content which they may have to teaching during Practice Teaching.

> The figure below shows the plan of a billiard table and the positions of the cue ball C and a red ball R.
>
> a) Draw a one-cushion path from the cue ball to the red ball. How many such paths are there?
>
> b) Draw a two-cushion path from C to R. How many are there?
>
> c) Can you draw a three-cushion path from C to R? How many are there?
>
>
>

These problems are just two examples of the type of investigation which allow the solver to use concrete, pictorial and abstract representations and appeal to the cognitive, affective and physical domain. One prospective teacher remarked after struggling for a couple of days, "I took four years of University math but this is the first time that I have ever done any real Mathematics".

APPENDIX E

CONTRIBUTIONS FROM

PANEL GROUP X

Words triggered by Images: Images triggered by Words.

A Paper

based on a presentation

of the same title

made at the Annual Conference

of the

Canadian Mathematics Education Study Group

(CMESG)

BY

A.J.(Sandy) Dawson

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Edmonton, Canada

June, 1981

"Words triggered by images: Images triggered by Words"

Introduction: Let me indicate straight away that what I am about to say is probably not new. Many others writing in maths education and other fields have,or are dealing with,this topic. Nonetheless, I sense that the time is ripe for a close examiniation of this topic vis-à-vis maths education.

My objective today is to briefly present to you the bare bones of some ideas, and to then invite you to engage in some activities. It is my hope that the activities will provide the impetus for us to examine our experiences, and to reflect about what our experiences have to tell us with respect to

a) how each of us generates mathematics;

b) how, if at all, the generation of mathematics involves not just the intellect, but also one's emotions or affectivity, and one's physical body; and

c) how we might structure some activities for students which would enable them to generate and learn some mathematics.

Let me begin, then, by providing you with the bare bones of what I've titled, "Words triggered by Images: Images triggered by Words."

In a paper I presented at ICME IV in Berkeley, California last August,[1] I began to explore the connections between Imre Lakatos' work, best known to most people through his book Conjectures and Refutations,[2] and the work of Caleb Gattegno,[3] founding Chairman of the ATM. What I wish to do here is to extend some of the ideas I introduced there, and to seek your response to those ideas.

I want to talk about images and words, images and language. I want to open for discussion the function of image-making -- imagination -- in the teaching and learning of mathematics. I wish also to examine, with your help, the relationship of imagination to language, especially the language of mathematics.

You may be wondering what all this has to do with Lakatos and Gattegno. Let me dwell on that for just a moment.

Lakatos proposed that a proof-and-refutation strategy was fundamental to the process of generating mathematics. Gattegno has argued that the powers of the human mind are indispensible tools for the learning of mathematics. Both men suggest that insightful guessing or intuition is an integral part of both the creation and the learning of mathematics.

Gattegno further suggests that even very young children possess powers of mind which teachers can use in assisting their students to learn maths. Moreover, one of the powers of mind which stimulates intuition or insightful guessing is that of imagination or image-making.

The fact that the power of image-making is crucial to the learning of maths was brought home to me with ego-crushing resoundness some years ago. The first time this happened to me was when I was taking a course in Non-Euclidean geometry. I simply could not build an image for myself as to what, for example, a Lambert quadrilateral would look like. I could not get a "feel for" or a "sense of" these rather strange shapes, or so they seemed to me. I quickly shifted to a study of number theory and topology.

Now, number theory I could get a grasp of, but topology was a different matter. There again, I could not create images in my mind of what topology was all about. Yes, I could "learn" the theorems. I could "do" the tests. I even passed the topology course with an excellent mark, but I knew -- deep inside -- that I didn't have the foggiest notion as to what topology was all about, because I didn't have the necessary images with which to work.

These were my own limitations. As a learner, I was stymied because I could not get a picture, so to speak, of these two fields of mathematics. My images were faulty or non-existent. My teachers had their images of these fields of study, but they could not transmit their images to my mind, try as they might. A learner, in this case, me, has to work with his own images. That is the crucial point: learners must create and work with their own images. Teachers cannot do it for them.

Nor can teachers transmit their images to the minds of their students, at least not in any sort of reliable fashion. Teachers must work with the images their students possess, however faulty or incomplete or even non-existent those images might be. Let me quote from that paper I gave at Berkeley:[4]

> What Gattegno has argued is that to tap children's minds to do mathematics we have to ensure that things like points, planes, and lines in geometry, for example, "... exist in the minds of ... students at the level of intuition, of imagery.[5] We need "... to give them the dynamics, and then let them talk"[6] about what they saw. Then, their "... words will be triggered by the images, and later the images will be triggered by the words.[7]
>
> So, first, we must have the images, the students' images, then we must have the children talk about their images, and only then should we attach words -- the conventional mathematical terminology, if you will -- to those images. Then, perhaps, we can formalize it all so that in the final product, the words trigger the images. What is being said is that the meaning should precede the words, the experiences precede the formalization. If this is done, then later, the words, the formalizations, will

> activate the mind to re-collect the experiences and the meaning.
>
> When one says -- and many do -- that the teaching of mathematics should be meaningful, I agree, but the meanings must be those of the children, and those meanings need not necessarily be based on real-life situations, so-called. The images of the mind are also real, and they are often richer than real life. Meanings can be constructed from such images.

In this recent book, Mindstorms, Seymour Papert[8] argues in a similar vein when he states that "... what a person can learn, and how he learns it, depends on what models (images) he has available." Papert goes on to say that "... anything is easy if you can assimilate it to your collection of models (images). If not, anything can be painfully difficult." Gattegno would contend, however, that the new integrates the old, thereby changing the old, rather than the old assimilating the new.

Be that as it may, what Papert does suggest is that computers can provide learners with a tool for checking their images -- and their intuitions -- by responding to the learners' conjectures. Lakatos certainly promoted "guessing" as a means of exercising one's intuition, and he also encouraged the strenuous checking of one's guesses or intuitions.

After 20 years in the education business, it is with some sadness I report that I have not seen very many classrooms where teachers actually worked with the images children create, or where children were encouraged to use their intuition to guess at an answer.

Teachers still do not seem to realize and/or feel comfortable with the fact that mistakes are a necessary part of learning. As Ken Iverson[9] said two nights ago, "a useful education is acquired by making mistakes, and having the courage to do so."

Well, let's move on and do some imaging. Let's use our intuition. Let's make some guesses. Let's make some mistakes.

I have a few activities I invite you to engage in for a while. After doing the activities, I would like to say just a few more words about language and images, and about the magic of communication.

Some of the activities I will present may be new to a few of you, and may therefore seem rather strange, especially for a group of mathematicians and maths educators. Consequently, I invite you to suspend your judgement for a half-hour or so about

a) the value of such experiences, and

b) the relevance of such experiences for learning/teaching maths.

The value and relevance will become evident as we go along, or at least that is my hope.

[The following three experiences were willingly engaged in by the twenty participants. Much discussion followed each activity, but this discussion was not recorded. However, I have tried to recapture some of the points that were raised, adding additional comments of my own where they seemed to relate directly to the topic of the talk. A.J.D.]

I. Sit back in your chair. Close your eyes if you wish. Note the points of contact of your body with the chair and the floor. Shift your body about until you feel comfortable and relaxed. Let any locked-up energy slide from your body down through your legs and toes and out onto the floor. Be at peace.

[PAUSE: Readers are invited to join in right now. Get yourself relaxed and comfortable, and then try this.]

1. Imagine a lemon.
2. Rotate the lemon.
3. Move the lemon to the right, and then to the left.

   [Pause]
4. Break the lemon into two pieces.
5. Rotate each piece.
6. Rotate each piece at right angles to each other.

   [Pause]
7. Rotate the pieces in opposite directions.
8. Put the two pieces together again.

   [Pause]
9. Seal the two pieces together so that you have the original lemon once again.

What did you experience?

Which senses did you use in creating your image?
(Could you smell your lemon, taste it, feel it?)

How did you break your lemon?

How did you seal your lemon together again?

Did the image you created remind you of previous experiences with real (?) lemons?

Who created your image?

[Participants readily volunteered their reactions. It seemed clear that everyone had a different image, and that each image was self created. My words triggered each participant to create their own lemon. Once created, the instructions were interpreted in many different ways. For example, some participants stated that they could not "break" their lemon -- they had to use a knife -- even though the lemon was imaginary! Because of the oblique way he split his lemon, one participant could not rotate his lemon.

It seems clear that each of us creates our own images even though we all may hear the same instructions. The leader (teacher?) cannot put images into learners' heads or minds. Moreover, there also seemed to be an emotional aspect to the images, because some participants said that the activity evoked memories and sensations long since forgotten. Affectivity is indeed active in all learning. A.J.D.]

II. Alright, relax yourself once again. Let's work with your lemon a bit more.

1. Once again, create your lemon but this time make it perfectly spherical.
2. Mark the surface of your lemon with a horizontal line about the middle of the lemon.
3. Mark two more lines on the surface of your lemon so that these two new lines are perpendicular to the horizontal line.

[Pause]

Keeping the image you have created firmly in your mind, answer these questions.

a) What shapes do you see on your lemon?

b) How many different shapes are there on your lemon?

c) Are any of these shapes congruent one to the other?

[Pause]

What did you see?

What did you create?

Are there any mathematical propositions or theorems or conjectures you could make, based on your image?

[This activity proved to be more difficult for some participants. They testified that they could not draw or mark the second and

third lines on the lemon -- their image dissolved. Other participants "saw" some triangles, while a few folks "saw" six sets of congruent triangles. Words do not create images: people do! A.J.D.]

III. This final activity was inspired by a lecture John Mason, of the Open University of Britain, gave at Simon Fraser University last fall.

Once again, relax.

1. Imagine a perfect circle.
2. Partition your circle into four congruent pieces.

   [Pause]
3. Do this in at least 3 different ways.

Come to the chalkboard and draw your image.

What conjectures can you make?

Can you prove your conjecture?

[Many participants reported that steps 1 and 2 were easy, but that step 3 caused them to stop cold, concluding that it was impossible to partition the circle in more than one way. The diagrams on the chalkboard persuaded these folks that the "impossible" had been achieved. However, a discussion then arose as to what was meant by "3 <u>different</u> ways." Time did not allow for this discussion to be concluded. Participants were left with the challenge -- as is the reader -- to explore this problem, using, instead of a circle, a square, a regular pentagon, a regular hexagon, and so on. A.J.D.]

<u>Conclusion</u>

What is being suggested here is that teachers must work with their students' images; not with the images they wished their students

had. Teachers must then encourage their students to talk about these images. From their talk and their images, teachers can assist students in verbalizing the meanings the students give to their images, and where appropriate, attach or supply the conventional mathematical terminology. This terminology, always subject to revision when new meanings arise, can then be used to recapture the images and meanings which gave rise to the words.

Bill Higginson of Queen's University, Kingston, has suggested that "renaming is remeaning." There is only one thing wrong with this phrase. It is backwards. The meaning always precedes the words. One only has to observe children learning to speak to note this fact. In the beginning there are no words, but there are certainly meanings, and, I would conjecture, there are rich images. The images and meanings come before the words.

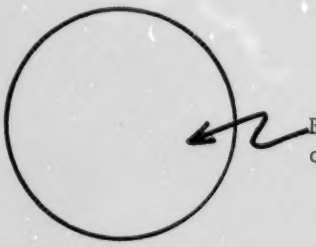
That is not to say that the images and meanings won't be refined, modified, perhaps even discarded, but before new words appear to replace the old, or before old words take on new meanings, new images and new meanings must be present. It is almost magical how new images generate new meanings which, in turn, are labelled with words.

Insofar as possible, we do not want students of mathematics to view the subject as being magical. However, we can never remove the magic completely. Let me elaborate on that point just a bit.

A few years ago, a set of books appeared, authored by Bandler and Grinder, titled The Structure of Magic.[10] These were not mathematics books, nor were they books about education. Yet they have, I believe, profound implications for education and the teaching

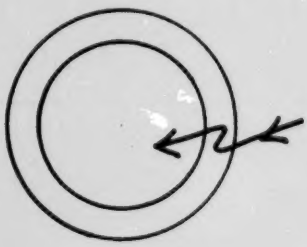
of mathematics. They are books about human beings, and how humans verbalize what they know.

The essential point of the books is that each human being, each of you, has within you an almost infinite amount of knowledge and know-how, all of which has intellectual, emotional, and physical aspects.



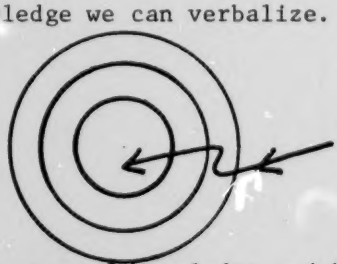
Everything a person knows or knows how to do.

The amount of that knowledge and know-how of which any of us is conscious is considerably less than the total amount we know.



Amount of knowledge and know-how of which we are conscious.

Moreover, even the amount of knowledge and know-how we are aware of, however small that may be, is considerably larger than the amount of that knowledge we can verbalize.



What we can verbalize

Consequently, the amount of knowledge and know-how about which we can verbalize is very small in comparison to all that we know. It is indeed somewhat magical, if not miraculous, that as human beings we do as well as we do in our quest for new knowledge, and in our

attempts to verbalize that knowledge.

The point is that children also have a vast amount of knowledge and know-how -- a fact not given serious acknowledgement by adults -- and part of that is made up of images they have created to give meaning to their world. By encouraging children to verbalize about their images, we may begin to get an idea about their meanings, and the words they attach to these meanings.

Hopefully, from that point we, as teachers, can begin to make connections between their language and mathematics, and in the process educate their awareness, their knowledge, and their know-how.

It is a never ending-cycle, but one which leads to an ever increasing knowledge of self and of one's world. It is magical in some sense. That magic can never be completely removed, but one's knowledge and know-hows become increasingly rich as one works at removing some of the magic.

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3. Caleb Gattegno has written widely. Some books which would be of interest to educators are The Common Sense of Teaching Mathematics, What We Owe Children, and The Mind Teaches The Brain, all of which are available from Educational Solutions Inc., 80 Fifth Avenue, New York, N.Y. 10011.

4. Dawson, op.cit.

5. Caleb Gattegno, "The Foundations of Geometry," For the Learning of Mathematics, Vol. 1., No. 1, Montreal, 1980. P. 11.

6. Ibid.

7. Ibid.

8. Seymour Papert, Mindstorms, Basic Books,Inc., New York, 1980.

9. Ken Iverson, "Mathematics and Computers," paper presented at CMESG meeting, Edmonton, Canada, June, 1981.

10. R. Bandler and John Grinder, The Structure of Magic, Science and Behavior Books, Inc., Palo Alto, 1975.

'X' ARGUMENTS FOR THE CONSIDERATION OF LANGUAGE BY RESEARCHERS IN MATHEMATICS EDUCATION

(Notes for a Panel presentation on "Mathematics and Language" Fifth Annual Conference of the Canadian Mathematics Education Study Group, Edmonton, Alberta. June 1981.)

Dr. William Higginson

Queen's University

Overview

Introduction - Context - Communication - Content - Cognition

Introduction

"Caveats auditors" - Confessions of a Logophile - Russell and Whitehead on nitpicking and woolgathering - On meaning and interpretation, 'X' as 'ten', 'former', 'x-rated' - On the Hi C's; 'Co's, Cos and 'cos - Hardy's advice for ambitious academics.

Context

Local/Global distinctions; the human factor and the "world problematique" - Significance of the area; Cassirer's 'animal symbolicum'; universal language à la Leibniz, Descartes Freudenthal, Sagan.

Possible Relationships between Mathematics and Language - (as Venn diagrams).

I The Disjoint Model: Math is math and language is language and never the twain...current.

II Nothing Buttery or Imperialist Models

(a) All math is language (mathematics is just another

(b) All language is mathematics

III The Overlap Model

References: Cassirer, An Essay on Man; Waddington, Tools for Thought; Higginson, 'Language, Logic and Mathematics.'

Communication

(Man as Social Animal ... Co-ing)

Linguistic squeaks in the learning of mathematics as 'clues to black boxes'; egocentration, evolution, Fred Creamcheese and Clinical methods.

Classroom discourse and the many-body problem

References: Ginsburg, Children's Arithmetic; Higginson, 'Berry Undecided', 'Upstaging Piaget', 'Because Zero is Nothing'; Easley & Zwoyer, 'Teaching through Listening'; Erlwanger, 'Benny'.

Content

Symbols, Notation and Representation

$$45 - 5, \quad 4\frac{2}{7} - \frac{2}{7}, \quad 5z^2 - z^2$$

Music, Computers.

References: Cajori, History of Mathematical Notation; Howson, 'ICME IV Paper'; Preston, 'Language of Early Mathematical Experience'; Higginson, 'Frames', 'Symbols, Icons and Mathematical Understanding'; Whitehead, An Introduction to Mathematics; Iverson, Algebra, 'Notation as a Tool of Thought'.

Cognition

Intellectual Construction - Language and Thought

Meaning - Thom (ICME 2)

Interpretation - Pair et impair (father in a raincoat?)
Bathroom on a bicycle
Blowing up fish and balloons
Bees and Beads

Metaphor - Threeks
Stay there for the present

Images

Artifical Intelligence and Cognitive Development

References: Donaldson, Children's Minds; Skemp, The Psychology of Mathematics Learning; Sinclair, 'ICME IV Paper;' Higginson, 'Threeks, Rainbrellas and Stunks'; Lakoff and Johnson Metaphors We Live By.

APPENDIX F

CONTRIBUTIONS FROM

PANEL GROUP Y

Groupe de discussion Y: Les relations entre l'histoire et la didactique des mathématiques

Contribution de Louis Charbonneau (U.Q.A.M.) le 19 mai 1981

Depuis une dizaine d'années, l'histoire des mathématiques a retrouvé une place qu'elle avait plus ou moins perdue dans le monde des mathématiques et des mathématiciens. Les raisons et les causes de ce retour sont diverses, allant probablement de la mise en accusation des sciences par les mouvements écologiques à la contestation venant de l'intérieur en réaction à une pratique mathématique que certains trouvent désincarnée et asséchante. Ce retour de l'histoire des mathématiques a amené de nombreux chercheurs à s'interroger sur la place que pourrait occuper l'histoire dans notre enseignement. Cette préoccupation n'est pas nouvelle. L'histoire est omniprésente dans les manuels de mathématiques du XVIII^{e} siècle et du début du XIX^{e} siècle. Par ailleurs, au tournant du siècle, nombre de pédagogues américains se sont penchés sur cette question. Qu'il suffise de penser à David Eugene Smith qui publia en 1900 son The Teaching of Elementary Mathematics, dans lequel plusieurs chapitres portent sur l'histoire de la numération, de l'algèbre et de la géométrie et, en 1923-1925, son célèbre History of mathematics, destiné à être à la fois un manuel et un livre de références pour les enseignants.[1),2)] Plus récemment, cet intérêt pour un tel usage de l'histoire s'est manifesté en autre par la publication par le National Council of Teachers in Mathematics (N.C.T.M.) de son recueil Historical Topics for the Mathematics Classroom (Washington, 1969) et par la fondation du International Study Group on the Relation between the History and the Pedagogy of Mathematics (ISGRBHPH) en 1972. Lors de sa première réunion à Exeter en 1972, dans le cadre du Congrès international sur l'enseignement des mathématiques, les membres du groupe firent ressortir six avantages pédagogiques de l'utilisation de l'histoire:[3)]

a) The communication of the sense of mathematics as an activity of human beings in contrast to a ready-made structure.

b) A recognition of the relationship between mathematics and the history and culture of our society.

c) An increased perception of the inter-relationship between different aspects of mathematics (algebra, geometry, analysis, applied mathematics).

d) A recognition of the growing and changing nature of mathematics: that mathematics has many futures as well as a history.

e) The improved understanding of modern mathematics generated in students as a result of (a) to (d).

f) The improved motivation for students at all levels arising from the humanizing influence of history and the existence of a potentially infinite number of open ended problems which can be encouraged and generated in a developmentally taught course.

Ces avantages sont avant tout de nature incitative en ce qu'ils touchent à la perception qu'ont les étudiants des mathématiques. Comme l'a écrit à plusieurs reprises Phillip S. Jones dans ses articles [4], il s'agit ici de modifier l'attitude des étudiants face aux mathématiques. Je crois qu'il y a encore un travail à faire dans cette direction, nonobstant le fait que la grande majorité des publications parues jusqu'à maintenant sur les relations histoire-enseignement sont orientées dans cette direction.[5]

Peut-être à cause de ma formation d'historien, cet emploi de l'histoire ne me satisfait pas entièrement. L'histoire n'y est qu'un outil pour affiner un acte pédagogique ou pour suppléer à ses faiblesses. La nature de cet acte n'est pas vraiment modifiée par l'apport historique. L'histoire y est un peu un cataplasme. N'y a-t-il pas moyen d'extraire de cette masse de connaissances qu'on appelle histoire des informations qui permettraient non pas seulement d'ajouter à une pédagogie déjà existante mais surtout d'édifier une pédagogie sur de nouvelles bases? Utopie! me direz-vous. Vous avez sans doute raison. Vous ajouterez peut-être alors: Pourquoi rejeter du revers de la main ce qui a été fait jusqu'ici? Alors, vous répondrais-je, ce sera parce que je me suis mal fait comprendre. Mon but n'est pas de rejeter les avenues qu'on a déjà commencées à explorer mais il est plutôt d'avancer timidement sur un sentier qui semblait un peu délaissé.

C'est en préparant un cours d'histoire des mathématiques qui s'adressait à de futurs enseignants du secondaire que j'ai commencé à méditer sur l'idée, si chère à Piaget, de l'existence d'un parallèle entre l'évolution cognitive de l'enfant et l'évolution des systèmes de connaissances à travers l'histoire. La question que je me posais, et qui demeure actuelle, pourrait s'énoncer ainsi: Comment tirer de l'histoire des informations propres à construire des stratégies d'apprentissage ou des séquences développementales qui pourront par la suite faire l'objet d'expérimentations avec des élèves ou des étudiants?

Je ne peux apporter de réponse à cette question. Toutefois, j'aimerais vous donner deux exemples qui me semblent montrer que la recherche d'une réponse à une telle question a des chances d'aboutir à un résultat positif. Le premier exemple met en scène deux de mes collègues de l'U.Q.A.M., Nadine Bednarz et Bernadette Janvier, ainsi que moi-même. Le deuxième exemple me mettra en relation avec Claude Janvier.

D'abord le premier exemple: Le premier thème auquel j'ai appliqué la question que je viens de poser est la numération. Il s'agit d'un sujet sur lequel la documentation se trouve relativement facilement. Ainsi j'ai basé mon étude principalement sur deux livres: celui de Menninger, Number Words and Number Symbols et celui de Guittel, Histoire comparée des numérations écrites. C'était là un travail de clerc en ce sens que je travaillais seul, discutant peu avec mes collègues. Lorsque mes idées commencèrent à se préciser, je donnai quelques conférences afin de partager le fruit de ma réclusion temporaire. A la même époque, Nadine et Bernadette terminaient une recherche expérimentale sur la compréhension de la numération chez les enfants de 6 à 10 ans. Quelle ne fut pas leur surprise de m'entendre reprendre quelques-unes des conclusions auxquelles elles étaient parvenues par des voies toutes différentes. Voici leurs conclusions, telles qu'énoncées dans un article qui devrait paraître dans Educational Studies in Mathematics:[6)]

a) "The skills developed by school mathematics programs do not transfer to this situation (comparaison de nombres). Our analysis even suggests that these skills lead almost exclusively to the "digit-by-digit" strategy".

b) "The children attribute a meaning to hundreds, tens, and ones more in terms of order than in terms of groupings".

c) "The concept of borrowing is rarely linked to the idea of exchanging".

d) "One of the most important difficulty was the impossibility to handle simultaneously two different kinds of groupings".

e) "Pupils have difficulty to conceptualise numbers as entities. They only see symbols placed side by side".

f) "We observe (...) that 4th year pupils made use more systematicaly of computation than 3rd year pupils".

g) "Children have a knowledge of numbers which is on the whole associated with some verbal support".

Pour ma part, j'étais arrivé à certaines de ces conclusions:

a) Importances fondamentales de l'oral dans l'acquisition de la séquence numérique. Je suis arrivé à cette conclusion en remarquant que l'évolution des nombres parlés se fait selon un cheminement semblable dans diverses langues.

b) Le passage aux calculs écrits dans une numération écrite est difficile. En effet, qu'il suffise de penser à l'usage extensif de l'abaque et des jeux de mains.

c) Le travail avec des groupements de groupements constitue une difficulté plus grande que le travail avec des groupements simples. Pour s'en rendre compte, il suffit de considérer l'ensemble des systèmes de numération pour constater que les groupements s'y font rares.

c^{1}) Le principe de positionalité est un principe sophistiqué, il n'existe que quatre systèmes basés sur ce principe.

La coïncidence des conclusions me semble montrer qu'un travail dans la direction que je propose risque de porter fruit. Nadine et moi sommes actuellement à préciser nos idées sur les liens qui se sont établis entre leur étude

expérimentale et mon étude théorique et ce en vue de l'élaboration d'activités d'apprentissage. Déjà, on peut dire que l'oral, l'abaque et peut-être une symbolisation non positionnelle du nombre y joueront un rôle central.

Passons maintenant à mon second exemple. Celui-ci porte sur la notion de fonction. Il s'agit d'une recherche théorique que j'entreprends avec Claude Janvier. Avant de dire quoique ce soit d'autre, je dois préciser que nous en sommes qu'au début. Donc, je me limiterai à esquisser la problématique, sans chercher à être trop précis. J'en serais d'ailleurs incapable.

Le concept de fonction est un concept fort complexe. Il peut se manifester sous diverses formes: sous forme verbale, sous forme de tableau, de schéma, par des graphiques, par une ou des équations. La compréhension de ce concept repose aussi sur différentes composantes: rôle du déterminisme, rôle du temps, rôle de la notion de variable, rôle de la notion de continu, rôle de la perception des différents types de taux de variation (par exemple: linéaire, non-linéaire). Notre étude repose sur quelques principes que voici:

a) La notion de fonction est une notion complexe, nous venons de le dire. Mais l'histoire nous permet de voir évoluer cette notion et ce malgré , et avec, sa complexité organique.

b) L'écriture symbolique ne contient pas l'essence de la notion de fonction, et en particulier de la notion de variable. Chaque étape nouvelle, en sythétisant l'écriture antérieure, camoufle et voile une partie du sens de cette dernière. Il faut donc se pencher sur les diverses étapes de l'évolution du symbolisme associé aux fonctions pour en faire ressortir les éléments maintenant mis au rancart.

c) Ce qui nous intéresse vraiment, ce ne sont pas seulement les diverses périodes au cours desquelles le symbolisme est stable (comme

dans le cas de la science normale, dans le langage khunien). Notre intérêt réside aussi dans l'instabilité, dans ces périodes où plusieurs façons de penser se juxtaposent l'uneàl'autre sans se détruire mutuellement. N'est-ce pas là aussi le fonctionnement de plusieurs élèves en situation d'apprentissage?

d) L'histoire enfin nous donne accès à certaines clefs d'interprétation du comportement intellectuel d'un étudiant en situation d'apprentissage.

Partant de ces principes, nous concentrons notre attention sur trois points: le rôle du temps dans la perception de la notion de fonction, la manipulation des variables à une époque antérieure à celle de leur identification propre, et enfin, et surtout, la perception du continu chez les enfants.

Je me permets, en terminant, de dire quelques mots sur ce tout dernier point. L'histoire de la notion de continu est difficile à cerner. Pour s'en rendre compte, il suffit de parcourir les livres de Boyer sur le calcul différentiel et intégral ou sur la géométrie analytique, ou encore le livre de Dhombres, Mesure, nombre et continu. Dans un cours gradué en histoire des mathématiques, je traitais des relations entre le discret et le continu, entre la géométrie et l'arithmétique chez les grecs et au Moyen-Age. Nous travaillions ce thème depuis quelques semaines lorsque Claudine, une étudiante qui avait enseigné quelques années au secondaire, vint me trouver. Elle était fort surprise car elle retrouvait chez les philosophes du passé les mêmes raisonnements que ceux avancés par ses élèves pour soutenir que sur une droite, il y a un nombre fini de points ou un "très grand nombre, non infini, de points". Cette constatation permit à Claudine de mieux comprendre le bien-fondé des arguments de ses élèves. Cela lui a aussi révélé la perception du monde que pouvait avoir ces adolescents.

Notre quatrième principe trouve ici une illustration. Partant de là, on peut espérer être amener par l'histoire à comprendre les mécanismes qui permettent de passer d'une vision discrète de l'univers à une perception, ou l'acceptation, d'un univers continu.

Notes

1) "This work has been written for the purpose of supplying teachers and students with a usable text book on the history of elementary mathematics (...). The subject has come to be recognize as an important one in the preparation of teachers of mathematics and in the liberal education of students in college and high school". (Tome I, p. iii)

2) Dans sa thèse L'histoire des mathématiques dans l'ens[illegible]ement des mathématiques; historique et attitude des professeurs des collèges québécois francophones, Université de Montréal, 1974, Jean-Paul Collette a partiellement répertorié et résumé les publications sur ce sujet depuis 1980 jusqu'en 1972 (26 publications).

3) F. Rogers, Relations between the History and the Pedagogy of Mathematics, Sommaire du secrétaire du groupe II, International Congress of Mathematics Education, Exeter, 1972, p. 1. Extrait de la thèse de J.-P. Collette (p. 156) indiquée à la note 2.

4) Par exemple dans Jones, Phillip S, 1969.

5) Dans une bibliographie que j'ai faite en 1978 à partir de E.R.I.C., sur une cinquantaine de titre, presque tous avaient cette orientation.

6) Bednarz, Nadine, Janvier, Bernadette; The Understanding of Numeration in Primary School, à paraître dans Educational Studies in Mathematics.

Panel Group Y: The relation between history and the pedagogy of mathematics.

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READING OF MATHEMATICAL CLASSICS CAN BE REWARDING

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(Talk presented at a special session on Pedagogy and History of Mathematics at the Canadian Mathematics Education Study Group, Edmonton, June 8, 1981.)

§1. The speed at which Mathematics is proliferating leaves hardly any scope for introducing mathematical classics or source-books in university instruction. If mathematics is viewed as a cultural stream, we owe it to our students to let them have some acquaintance with some of the great mathematical masterpieces. The best time to do so is perhaps in the third or fourth year of a university mathematics program, when students will already have had some exposure to a wide spectrum of mathematics. It is by no means easy to make a selection of such classics, nor will there be agreement as to which works are suitable for the classrooms. The writings of Newton and Leibniz are certainly classics but not easy to handle in classroom situations. On the other hand the works of MacLaurin, L'Hospital, Johann Bernoulli on Calculus are eminently suitable as specimens of classic Calculus texts. Students will recognize here a wealth of information and appreciate how Calculus used to be done creatively in the years immediately following its invention. One cannot fail to be impressed by the daring freshness present in these works, a freshness not much in evidence in many of the latter-day sophisticated textbooks. Euler's Introductio in Analysin

Infinitorum (1748) is both a classic and a masterpiece of elegant exposition that is at once enjoyable and rewarding. Lagrange's Theorie des Fonctions analytiques (1797) is not only good reading but also rich in polemics, references, and suggestions. As a bridge connecting the era of inventive and creative mathematics of the Bernoulllis and Euler with the era of rigor ushered in by Bolzano, Cauchy and Weierstrass, Lagrange's Theorie des Fonctions hold an important place in History. Lagrange was unwilling to subscribe to the then incipient notion of 'limits' as the much sought after 'true metaphysic of the Calculus'. Lagrange sought a way to found Calculus without introducing infinitesimals then in disrepute and without reference to limits. We know now that his attempt was not successful. But his Theorie des Fonctions will always be regarded as a heroic enterprise. Cauchy's Cours d'Analyse (1821) is very instructive and yet full of freshness. Selected chapters from MacLaurin, L'Hospital, Johann Bernoulli, Euler, Lagrange and Cauchy will constitute a very important contribution to the mathematical education of our graduates, most of whom learn their Calculus and analysis from one or two specified textbooks or lecture notes prepared by teachers. Exposure to the greats in their final years will act not only as a corrective but as a life-long cultural acquisition. The above examples are from the Post-Newtonian era. Pre-Newtonian classics of Viete, Napier, Fermat and Pascal are also worthy of a place in

instruction. There is the problem of their accessibility in English. Perhaps the International Mathematics Union will see their way to publishing the great mathematical works of the 17 - 19 century as a series of monographs in English. I have in mind a series like the famous German language series of Ostwald's Klassiker der Exakten Wissenschaften.

§2. In this section I will give illustrations of some of the dividends my students and I gained from a reading of Euler, Legendre, Gauss and Cauchy.

A. Euler (1707-1783) Introductio (1748)

The chapters 6 and 7 of Vol. 1 of 'Introductio' are both charming and challenging. One begins to realize that it was Euler who introduced the tradition of treating logarithms as derived from an exponential function with any given positive base a. Euler goes into the matter with the zeal of a promoter. In fact, high school logarithms and frequently university logarithms still cling to this tradition. In Chapter 6, he gives illustrations showing the effectiveness of logarithms in calculating compound interest, population growth, etc. Chapter 7 is more analytic. He states there without any qualms that for an infinitely small quantity w, a^w is equal to $1 + kw$, where k depends on the base a. Then through a succession of naive but daring moves he establishes all of the elementary properties of a^x, e^x, etc. concluding with the two

formulae:

$$e^z = (1+\frac{z}{i})^i \quad \text{and} \quad \log(1+x) = i((1+x)^{\frac{1}{i}}-1)$$

where i is infinitely great.

Students marvel at the total lack of inhibition in Euler's Introductio. Euler's style is breezy and conversational. Playfully Euler proceeds from one step to another. Euler obtains a series expansion for a^x in the form

$$a^x = 1 + kx + \frac{k^2x^2}{1.2} + \frac{k^3x^3}{1.2.3} + \frac{k^4x^4}{1.2.3.4} + \ldots$$

without invoking differential calculus. Now k being a number associated with the base a, it is natural that one prefers to work with a base, the associated number k of which is 1. One is at once led to the number e, which is then

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots .$$

Generally, a and k are connected by

$$a = 1 + k + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \frac{k^4}{1.2.3.4} \ldots .$$

(Preceptive readers will notice that k is the derivative of a^x at $x = 0$, and $a = e^k$.)

Introductio is not a text on Calculus, but that of analysis known at that time by the name of Algebraic Analysis. [This kind of analysis was regarded as propaedeutic to Differential Calculus. In those days one did have to have

some knowledge of series representation of trigonometric, logarithmic and exponential functions before taking on Calculus.] Euler goes on to derive the formula,

$$\log_a(1+x) = \frac{1}{k}\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots\right),$$

where k is the constant associated with a. From this he obtains the explicit value of k as

$$k = \frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \ldots .$$

Euler was interested in $a > 1$, but did not disturb himself with questions of convergence. He does not hesitate to claim that the value of k associated with 10 must be equal to the value of the series

$$\frac{9}{1} - \frac{9^2}{2} + \frac{9^3}{3} - \frac{9^4}{4} + \ldots . \qquad (*)$$

Earlier he used the formula $a^w = 1 + kw$ to obtain $\log_a(1+kw) = w$. Bu taking $kw = \frac{1}{10^6}$, he uses the Briggsian common logarithms to obtain $\frac{1}{k} = .43429$ and hence $k = 2.30258$. Euler betrays some uneasiness with the above series (*), as he observes:

"It is difficult to appreciate that 2.30258 is equal to the series (*), for the members of the series increase in value of the series in the usual computational way. This unpleasantness can, however, be circumvented by means of a procedure to be given soon."

Euler than develops another series expansion for k,

$$k = 2\left(\frac{a-1}{a+1} + \frac{(a-1)^3}{3(2+1)^3} + \frac{(a-1)^5}{5(a+1)^3} + \ldots\right) .$$

This expression is more comfortable and in the case of a = 10

$$k = 2\left(\frac{9}{11} + \frac{9^3}{3.11^3} + \frac{9^5}{5.11^5} + \frac{9^7}{7.11^7} + \ldots\right) .$$

When reading Euler, students see how the different results relating to logarithmic and exponential functions hang beautifully together. Euler notes that the natural logarithms are also called hyperbolic logarithms 'because the area of a hyperbola can be calculated with these logarithms'. The chapters 6 and 7 of Euler's Introductio are very helpful in obtaining a well-rounded view of exponentials and logarithms with arbitrary basis. In the light of latter-day knowledge one is forced to the thought of using the sequence $(1+\frac{x}{n})^n$ as a more acceptable basis for treating exponentials, or, for that matter using the sequence $n(a^{1/n}-1)$ as a basis for logarithms. The latter is pedagogically simpler, though less favoured.

Chapters 8-11 deal with circular functions. They contain a wealth of new information. The formulas $e^{ix} = \cos x + i \sin x$, $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ are found here. Sinx and Cosx are developed as infinite products (Euler's original discovery!) which are then exploited to find the values of $\sum_{n=1}^{\infty} \frac{1}{n^s}$, s = 2, 4, ... 26. In particular, the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is obtained to be $\frac{\pi^2}{6}$. These two chapters are perhaps more suitable at the graduate level.

B. A.M. Legendre (1752-1833)

Alternative proofs of Fermat's theorem $a^p = a(\text{mod}p)$, and Wilson's theorem $(p-1)! = -1\ (\text{mod}p)$ are always welcome. Modern textbooks establish Fermat's theorem first and then Wilson's theorem. When reading Legendre's 'Theorie des Nombres' with two of my students, the following idea occurred to us.

> If $(x+1) \ldots (x+p-1)$ is developed as $x^{p-1} + a_1 x^{p-2} + \ldots + \ldots A_{p-2}x + A_{p-1}$, and p is prime, then $A_1, \ldots A_{p-2}$ are divisible by p.

A few examples seemed to confirm this. If the above idea is correct, then one obtains both Wilson's and Fermat's theorem at one stroke. My pleasure was not diminished by my later knowledge that Lagrange's proof of Wilson's theorem in 1771 was founded exactly on this idea.

Legendre chose to use the following remarkable formula:

$$1.2.3 \ldots m = m^m - \frac{m}{1}(m-1)^m + \frac{m(m-1)}{1.2}(m-2)^m + \ldots .$$

Legendre does not give proof of it. He simply mentions that the formula is obtainable from the theory of difference equations. Students do not see difference equations in their mathematics any more. My students felt invited to delve in the elements of this topic. That was by itself a reward for me.

C. C.F. Gauss (1777-1855) Disquisitiones Arithmeticae (1801)

i) The first four sections of Gauss' Disquisitiones Arithmeticae are rewarding. We see Gauss develop his thoery of congruence of numbers (mod n), and establish the Fundamental Theorem of arithmetic. He does not use or mention even for once the Euclidean Algorithm for his treatment of G.C.D. (Greatest Common Divisor) and L.C.M. (Least Common Multiple). Gauss perceived that the use of additive properties of integers is not very relevant for the multiplicative theory of integers. Led by Gauss' suggestion we were tempted to take the notion of L.C.M. as more basic than G.C.D. and found that the properties of both L.C.M. and G.C.D. were more easily obtained in this way without straying too far from the definitions. A set of necessary and sufficient conditions was obtained for a commutative cancellation semigroup with unity to have the property that any two elements possess an L.C.M. (and hence G.C.D.).

ii) In articles §§52-54, Gauss proves the existence of a primitive element in the multiplicative group of non-zero elements of the field z_p (p, a prime). His proof is easily exploited to obtain the following generalization:

> A finite group of order n having the property that for all divisors d of n, there is almost one subgroup of order d, then G. is necessarily cyclic and hence commutative.

This theorem, though not new, is seldom noticed in the textbooks on Algebra or even in textbooks in Group Theory. From this theorem, it is but a short step to proving the famous result in the theory of finite fields, namely, that the multiplicative group of a finite field is cyclic. For me raised on the tradition of Albert, Birkhoff-MacLane and Artin, this was indeed a pleasant experience and surprise.

iii) Seeing Gauss establish the property of Euler's totient function $\phi(n)$, namely, that for any positive integer n, $n = \sum\phi(d)$, where d ranges over all divisors of n, one cannot escape the suggestion that perhaps the above property characterizes the function $\phi(n)$. I had not seen any suggestion of this before or since. It turns out that this property does indeed characterize Euler's function $\phi(n)$. This was a small but nice reward from Gauss' D.A.

D. A.L. Cauchy (1789-1857)

Cauchy's Cours d'analyse (1821) is good reading even by today's standard. The discussion on the four fundamental functional equations is superb. Analysis had indeed come to maturity in the hands of Cauchy. After reading Euler, students notice the difference in temper when reading Cauchy. The age of rigor is seen to be in. Cauchy's claim that the sum of a series of continuous functions when convergent converge to a continuous function is found in this book. This claim

was not questioned until Abel (1826) produced a counter-example. Students are reminded that the additional notion of uniform convergence is important to uphold Cauchy's claim. But this notion was not yet available. The experience of seeing great masters occasionally nodding or tripping is instructive.

Conclusion

I have tried to give just a few examples from a selection of four great classics. Other great works, say Lagrange's work on the theory of equations, or Gauss' Theory of Surfaces are also suitable. For students reared on abstract and rigid pure mathematics, a selection of writings from Galileo, Huygens, Newton, the Bernoullis, Lagrange, Laplace and Hamilton on mechanics can be refreshing. No matter how varied the fields of applications may be, it is undeniable that the greatest victories of all times were scored in mechanics.

Source books exist but they do not have the same effect as reading in extenso from the classics. The English speaking world has lagged behind other languages in the idea of having seminal works in mathematical works translated.

What should a teacher know about the history of mathematics?

David Wheeler CMESG/GCEDM 1981

1. The question cannot be answered without setting it in the context of all that it would be good for a teacher at high school or college level to know. (I omit consideration of elementary school teachers here.) Let us say: an educated, professional teacher "should know"

(i) a fair amount of classical mathematics
(ii) how to set about learning some more
(iii) how to use some mathematics in a non-routine fashion
(iv) how to function as a mathematician, i.e. to be mathematically creative at some level, however elementary
(v) "about" the historical, philosophical, psychological, sociological and cultural aspects of mathematics

And all this in addition to a professional grasp of didactics and pedagogy!

Since their mathematical and professional education hasn't given many teachers all these knowledges and know-hows, we have to address priorities. I cannot doubt that (ii), (iii) and (iv) are more important for the teacher and the students than "knowing about" the history of mathematics.

Perhaps, though, several birds can be killed with the same stone. It might be possible for some mathematics to be learned from a historical perspective so that the mathematics and the history were learned together. Yes, it might. But it appears very difficult to do. Toeplitz' Calculus, a genetic approach, for example, is an interesting sourcebook but a poor textbook. (Admittedly the book as we have it is incomplete and unrevised.) A recent attempt by Priestley to do a similar job is slow-moving in the extreme - a nice "liberal arts" course, but not much good for prospective mathematicians.

A constraint, we may note, is that both instructors and students in universities, at undergraduate level, expect to be able to construct a whole course from a single book, making almost impossible demands on the textbook writer, however enlightened. The best he can do is to provide a number of historical comments or asides which will probably be ignored. But why should a mathematics course not be based on more than one book? An analysis course, say, with Spivak and Borel as set books, and associated readings from Cavalieri, Fermat, Leibniz, Riemann and Dedekind, etc., would be worth a fourth to a third of an honours programme in pure mathematics.

There are primary source materials in university libraries. Should not every undergraduate in an honours programme be required to work up at least one short topic from them?

Do undergraduate instructors advise their students to read some secondary source materials, books "about" the history of mathematics? Should it be left entirely to the student's individual initiative to spend some of the available time in background reading, as a student in most arts subjects would expect to have to do?

2. The questions about undergraduate mathematics programmes have their counterparts in the school and college situations, where they are generally even more difficult to answer.

At high school level experiemnts have taken place in introducing the study of original documents in the teaching of science. But most of the documents of relevance to high school mathematics either do not exist, or are untranslated, or are expressed in arcane language and notation. One could, perhaps, use Newton's proof of the binomial theorem, the Pascal-Fermat correspondence to introduce probability, Archimedes on some mensuration problems. But even these relatively accessible and uncomplicated accounts present conceptual and interpretative difficulties. And, having studied them, would the students' mathematical and historical gains be significant?

It does not seem appropriate to recommend high school students to read "about" the history of mathematics when they have probably not yet developed much of a historical sense. I don't find the historical "anecdotes" that sprinkle the pages of some texts objectionable, as some purists do, but they contribute very little to historical understanding. Howver it may not be a bad idea for students to learn a few names and a few dates even if these only serve as convenient markers that will acquire historical significance later.

What, then, can the high school teacher do?

(a) Relate some anecdotes (but try to get them right).
(b) Present some historical "capsules" at appropriate points in the curriculum. (The term is from the NCTM Yearbook, which gives suitable examples.)

and, more ambitiously,

(c) Raise and follow up questions. Not only "Who was Pythagoras?" (which usually gets answered by folklore rather than fact) but, say, "How did the trigonometric functions get their names?", "Why are statute miles different from nautical miles?", "Does e stand for exponent or for Euler?", and so on. Starting on one of these questions can be like pulling a thread which will lead back into history, but the occasion is provided by some familiar item in the standard curriculum.
(d) Prepare a topic in the form of a detailed "research brief", integrating questions about the historical origins of the topic with mathematical questions for investigation by the students. The booklets written by Dick Tahta, and distributed by the Leapfrogs Group, provide some possible models.

The resources which would help the high school teacher do all these things are in general non-existent or inaccessible. But supplying more resources, though important, will not by itself effect any change. The wealth of material written for teachers in Germany could readily be translated but, as things are, wouldn't have a market. The educational traditions are completely different from our own. The biggest obstacle to the introduction of a historical component into the teaching of mathematics in N. America is the general lack of awareness of mathematics as a cultural force.

3. Perhaps the only argument likely to win adherents in N. America would be the claim that a historical perspective improves the mathematical performance, competence and understanding of students. It is hard to sustain such an argument convincingly.

Two of the arguments that are occasionally used seem weak - history as motivation and as humanising the study of mathematics. In the first case, the motivation, if any, displaces students' attention away from mathematics on to its history; in the second case, direct rather than proxy humanisation would be more effective - i.e. giving students the chance to work out some mathematics arising from a problem that interests them here and now.

To get at the true relation between the pedagogy and the history of mathematics requires going deeper. The common ground between pedagogy and history is that both are concerned with mathematics in the making, in the process of construction, in human minds.

The notion that pedagogy can derive benefit from history has often been put forward - in a weak form which says that history can show up the conceptual difficulties of mathematics, and in strong form expressing itself in a version of Haeckel's Law ("ontogeny recapitulates phylogeny"). Polya and Piaget acknowledge the power of some "genetic principle", though neither takes the extreme position of Haeckel's Law - as, indeed, who could? The questionable feature in any form of genetic principle applied to pedagogy is that it accentuates the "biological" factors in learning and suppresses the cultural factors (a mistake the Russians, for obvious reasons, do not make). Wilder has taken a different direction by looking at mathematics historically as the product of an interaction between "internal" (i.e. mathematical) forces and "external" (i.e. cultural, ideological) forces. It doesn't appear to be quite the right mix since his internal forces don't seem to cover genetic factors at all, but his suggestion that the historian of mathematics might use techniques akin to those of the anthropologist is a productive idea.

A good deal of mathematical history has been assembled and written by people who were essentially mathematicians and not historians. With the exception of a few giants like Heath, many of them now seem to have been historically naive and amateurish. So it would be a mistake to think that mathematical history is all "there" and our only preoccupation is with whether and how to draw on it for educational purposes. Klein and Szabo have recently revised some commonly-held opinions about Greek mathematics. As time goes on we shall certainly find that a lot of the history we think we know will prove inaccurate in detail and misleading in general character.

Finally, the common ground referred to above would enable the relation between pedagogy and history to be reversed. Peple whose pedagogical experience has taught them a lot about how mathematical knowledge is constructed in students' minds could use their awarenesses to ask the historians questions that no one else has thought to ask.

APPENDIX G

CONTRIBUTIONS FROM

PRODUCTION/AD HOC GROUPS

Mathematics in China

by

Tony Thompson

Briefly, I might say that because I lived in China for 11 months (10 in Nanjing and one in Tianjin), taught at two Universities, had students from all over China, attended a mathematical conference in Suzhou, visited other institutions, and, above all, had three children in local chinese schools (primary and secondary) I think I did get a reasonable overall picture.

I thought the best way to convey this picture would be to try and describe the life of a mathematician in present day China. This will not, cannot be the life of any actual mathematician for those active now had their early education some time ago when conditions were very different. And those young now will grow up into a China that is likely to be much different. For, if one thing is true of China, it is the rapidity with which certain aspects of life change.

So: a future mathematician born now is likely to be born in the city rather than the country. For, although 4/5 of China's population live on farm communities, most of the intellectuals I met had been born and bred in the city. He (I will say he because, as here, there are presently few female mathematicians -- in my class of about 45 there were 4 women) will have at most one brother or sister. There is very rigid population control in China at present. His parents, for the same reason, will not be young. Both will be working and the likelihood is that both will have intellectual sorts of jobs.

Because both parents work his early formal education will be in a nursery and then a kindergarten where he will learn to count and to read and write the simplest characters. This is followed at age 6 with elementary school. Formal schooling will be 10 years - 5 in each of primary and secondary. Throughout he will be in large classes (my son's class was a Grade 1 class of 48 and the other children in secondary schools were in classes of 60-70) which are, therefore, taught in a formal way with strict discipline. In primary school the 3 R's are emphasized. Arithmetic is done with the aid of an abacus from

Grade 3 and he will become very proficient with this. In Grade 3 he will compute with large numbers and use all four operations. He will also begin to use decimals in Grade 3 and, a little later, fractions. As an aside one might point out here that the systems of currency, weights and measures have all been decimal for a long time (since long before Napoleon).

Here are some typical exercises from the Grade 3 text. (I say the text because there is only one in use throughout China in a very uniform system.) It is a copy of p. 59 of that book:

4. 46 × 98 99 × 157 8500 ÷ 50
95 × 102 4680 × 5 4848 ÷ 202
8 × 425 × 125 × 4 2000 - 178 - 922

5. 2130 ÷ 30 13156 ÷ 26 13900 ÷ 600
3718 ÷ 59 73634 ÷ 18 62582 ÷ 548
9020 × 80 20731 - 784 397440 ÷ 192
512 × 625 10700 × 340 3008 × 450

(These problems are to be done either on paper or with the abacus - whichever the student prefers.)

6. Find x

x × 31 = 6448 403 × x = 17329
x ÷ 610 = 287 68 × x = 72964
72.8 + x = 10.24 x - 17.2 = 58.7
53.14 - x = 6.37 8 × x = 15.36

And here are some harder problems from the same text (p. 52).

9.
```
   8 -
 × - -
 -----
   - -
 - -
 -----
 - - 8
```

```
          × ×
      -------
  × 7 | 14 × ×
      | ×× 5
      -------
          × ×
          × 1
          ---
            0
```

p. 115 Fill in the blanks with the largest possible whole number.

( ) × 24 < 100 46 × ( ) < 217
75 × ( ) < 620 ( ) × 53 < 302

Thus, in those first five years of school the future mathematician will become highly proficient at all kinds of computation. He will also work hard. Primary school (like secondary school and University) operates on a 6-day week and a 10-month year (2, 5-month terms) and he will have quite a bit of homework each day though this will be more writing than arithmetic. Chinese characters need a lot of practice to perfect.

Secondary school will continue this education for a further 5 years (though probably soon to be extended to 6). By the time he has finished he will know algebra, trigonometry, Euclidean and coordinate geometry, logs, exponentials, (log tables are used for calculations - no calculators yet and no slide rules in school), sequences, series, limits (no Calculus). It reminded me of an English Grammer school curriculum of 30 years ago. The work is not so routine here. Many quite difficult problems which offer a real challenge to the budding mathematician's ingenuity. Again, lots of homework.

The high school schedule:

6 days/week	7:20 a.m.	Arrive	
	7:30 a.m.	Morning Exercises	
	7:50 a.m.	1st class begins	4 classes
	11:30 a.m.	Lunch	
	1:50 p.m.	return to school	
	2:00 p.m.	1st class beings	
	3:40 p.m.	School ends M.W.F.S.	
	4:20 p.m.	School ends T.Th.	

There is some 'streaming' into Science or Humanities. The following is the timetable for 4th year of the Science class:

M	T	W	Th	F	S
MATH	PHYSICS	CHINESE	ENGLISH	CHEMISTRY	MATH
PHYAICS	MATH	PHYSICS	MATH	PHYSICS	POLITICS
CHINESE	ENGLISH	MATH	CHEMISTRY	CHINESE	ENGLISH
HISTORY	CHINESE	ENGLISH	PHYSICS	CHINESE	CHINESE
PHYSICAL TRAINING	CHEMISTRY	POLITICS	HISTORY	MATH	STUDY
SELF STUDY	MEETING (School or class)	SELF STUDY	PHYS. TRAINING	STUDY	GENERAL CLEANING
	SPORTS		STUDY		

Thus, in total:				
MATH	6 classes	CHEMISTRY	3 classes
CHINESE	6 classes	HISTORY	2 classes
PHYSICS	5 "	POLITICS	2 "
ENGLISH	4 "	PHYS. TRAINING	2 "

Examples of exercises in 3rd year of secondary school:

Quadratic equations are not always completely straightforward:

$$\frac{2x}{x-1} - \frac{1}{x+3} = \frac{3}{x-1}$$

$$\sqrt{2x+3} = x$$

Nor are simultaneous equations in 2 unknowns:

$$\begin{cases} 2x^2 + 5y^2 - 4x + y - 6 = 0 \\ x^2 + 2y^2 - 2x + y - 3 = 0 \end{cases}$$

$$\begin{cases} x^2 + y^2 = 5 \\ 2x^2 - 2y^2 - 3xy = 0 \end{cases}$$

$$\begin{cases} x^2 + 2xy + y^2 = 9 \\ (x-y)^2 - 3(x-y) + 2 = 0 \end{cases}$$

In the 4th year the log questions can be quite tricky:
Evaluate (without tables):

$$\log_{10}(\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}}) + \log_{\sqrt{2-\sqrt{3}}}(2+\sqrt{3}) \quad .$$

If $y_1 = \log_a(x^2+1)$ $\qquad$ $y_2 = \log_a 2x$.

For what values of x is $y_1 = y_2$? $y_1 > y_2$? $y_1 < y_2$?

Given $a^2+b^2 = c^2$ prove $\log_{c+b} a + \log_{c-b} a = 2 \log_{c+b} a \cdot \log_{c-b} a$.

Secondary school culminates with a major hurdle - the University entrance examination. This is a national examination though different Universities set different levels for entrance. The student may list a number of Universities in order of preference. It is an extremely tough exam and of those finishing secondary school only about 2% go on to University (though others may go on to other forms of tertiary education -- teacher training colleges etc.). It is not uncommon for students to stay at home for a year or two after secondary school to attempt the exam one or two more times.

Our budding mathematician, presumably, passes the exam and goes to one of the major Universities where his course of study will be like the following (a recently approved programme adopted by the mathematics departments of Nanjing University, Fudan University and two or three others).

University Courses

Course	Total # of Hours	Hours/Week in Semesters 1	2	3	4	5	6	7	8
		[Weeks 17	18	18	18	19	19	18	9]
Political Education	68								
Political Theory	218	2	2	2	2	2	2		
Foreigh Language	284	4	4	4	4				
Physics	142	2	2	2	2				
Analytic Geometry	68	4							
Linear Algebra	157	5	4						
Analysis I	192	6	5						
Analysis II	144			4	4				
General Physics	180		5	5					
O.D.E.	72			4					
Abstract Algebra	54				3				
Theoretical Mechanics	72				4				
Complex Analysis	76					4			
Topology	57					3			
Stats/Probability	76					4			
Real and Functional Analysis	112						4	2	
Differential Geometry	57						3		
P.D.E.	76						4		
Electives (MATH)	314					4	4	6	6
TOTALS	2401	23	22	21	19	17	17	8	6

The electives are likely to be topics like Numerical Analysis, Number Theory, Optimization,...

As can be seen from the table, even allowing for the total of 288 hours spent in tutorials as opposed to lecture hours, the total time in class greatly exceeds the normal 4 year, 5 class/year Canadian total. The extent of the specialization in mathematics is also greater. Thus he completes his undergraduate training with a pretty broad, solid, if slightly old-fashioned knowledge of mathematics.

Entrance to graduate school is again by examination. Some of the present graduate students have not had the above training -- in fact during the cultural revolution some had very little training at all.

Assuming our mathematician successfully completes this course he will join a large University department, (at Nanjing about 100 - the total University faculty is about 2000 for a student population of about 5000). He will not be better paid than in industry (about $50 per month). He will almost certainly live in a University apartment close to the campus. His teaching load is likely to be quite light with time for research. At Nanjing there is an excellent library with good collections of journals from the West, from USSR, from China and Japan. It is housed in the mathematics department and looked after with care by a research mathematician.

Since the department is large it is likely to be organized in research groups -- at Nanjing an operator theory group, a numerical analysis group, an algebra group,... led by a senior mathematician who, most probably was trained abroad -- either in the U.S. in the late 40's or in the USSR in the mid-fifties.

It is difficult to know how to end this account. The senior academics do not retire. They remain active for as long as they wish and keep their positions and salary for their lifetime. At Nanjing such a one is Professor Zhen who was a student of Marshall Stone's in the 30's and who taught Functional Analysis in Chungking throughout the war. Like other older members of the society he is accorded considerable honour and prestige at Department and University gatherings.

I hope this account has conveyed something of the mathematical life of China which, with an extremely ancient tradition, seems set for new vigour with a strong infusion from the West.

LE DODÉCAÈDRE ET

ET

LA GÉOMÉTRIE PROJECTIVE D'ORDRE 5

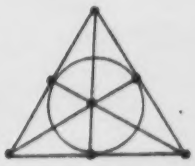
FERNAND LEMAY

UNIVERSITÉ LAVAL

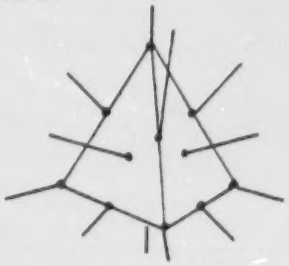
QUÉBEC, CANADA

§1. L'HÉRITAGE DES SOLIDES PLATONIQUES.-

La présentation de l'idée de *structure géométrique*, comme celle de toute autre notion ou système mathématique, doit s'accompagner d'illustrations; ne serait-ce que pour conjurer le risque de proposer une "théorie vide". Aussi fournira-t-on aussitôt quelques exemples de *structures projectives* et, sans doute, posera-t-on la question de trouver la plus simple des géométries. Surgit alors celle de *Fano* (ou géométrie projective d'ordre 2) qui, avec ses 7 points et ses 7 lignes (six segments et un cercle),



n'est pas sans évoquer une vue en élévation d'un tétraèdre. L'ensemble des 7 points correspond alors à la *gerbe des axes de rotation* du tétraèdre:



Grâce à cette coincïdence, le tétraèdre transmet un lot de transformations qui contribuent efficacement à la génération du groupe des colinéations de la géométrie de Fano (†). Après la découverte de cette heureuse coïncidence, on se devait de jeter un regard de routine sur le *cube* afin de ne pas laisser échapper une autre correspondance analogue. Surprise! La gerbe

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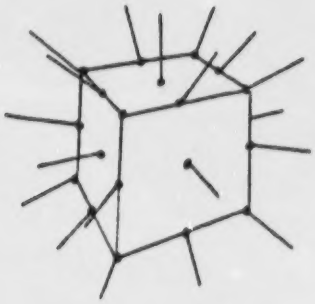
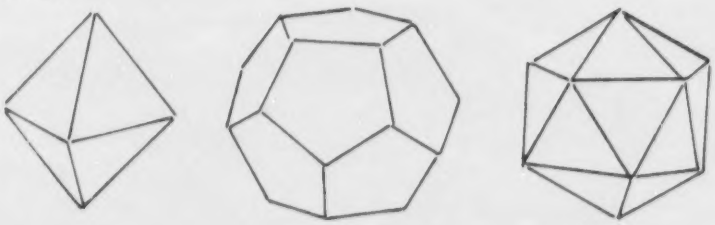
(†) *Voir nos deux courts articles, "Imagination dissidente" et "Motivation intrinsèque", parus dans le Bulletin de l'APAME, mars et novembre 1979.*

des axes de rotation du cube



comprend *13* éléments, c'est-à-dire exactement autant qu'il y a de points (et de lignes) en géométrie de *Young* (ou géométrie projective d'ordre *3*). Cette fois dans un contexte moins restreint l'hexaèdre offre, pour opérer en géométrie de Young, le groupe de ses 48 isométries; et nous avons eu l'occasion de raconter (†) comment *5* générateurs, essentiellement issus d'une rotation d'un demi-tour, parvenaient à engendrer le groupe des *5 616* colinéations de cette géométrie. Il était devenu urgent de nous tourner maintenant vers les autres *polyèdres platoniques* et dénombrer les gerbes d'axes de l'*octaèdre*, de l'*icosaèdre* et du *dodécaèdre*.



Quel étonnement de découvrir que les coïncidences observées au début subsistaient et que ces gerbes renfermaient, elles aussi, autant d'éléments que certaines géométries projectives: *13*, *31* et *31* respectivement, c'est-à-dire, pour ce qui est de la gerbe du dodécaèdre et de l'icosaèdre, autant d'axes qu'il y a d'éléments (de l'une ou l'autre espèce) en géométrie projec-

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(†) *Voir "Le chant de l'hexaèdre", Université Laval.*

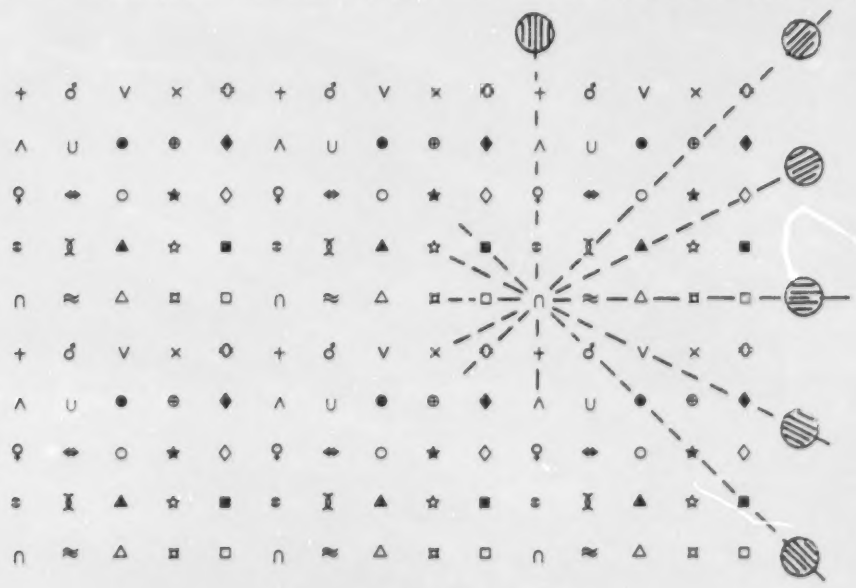
tive d'ordre *5*. Et nous nous sommes souvenus que les cinq polyèdres platoniques sont inscriptibles deux à deux les uns dans les autres: le cube dans l'octaèdre ou réciproquement, le dodécaèdre dans l'icosaèdre ou réciproquement, le tétraèdre en lui-même, et que c'était là la raison par laquelle nous nous retrouvions avec trois cas plutôt que cinq.

Un lien plus ou moins fortuit paraissait donc exister entre les polyèdres platoniques et les géométries projectives d'ordre *2*, d'ordre *3* et d'ordre *5*.

§2. TROIS MODÈLES.-

Restait à traiter le dernier cas. *Comment aborder la géométrie du dodécaèdre?*

Au cours des expériences antérieures avec le tétraèdre et l'hexaèdre nous avons naturellement eu recours aux *"plans cycliques"* déduits de Z×Z par identification des points dont les coordonnées (abscisses d'une part, ordonnées d'autre part) différaient d'un multiple d'un nombre naturel donné *n*, ce qui conduit, comme on sait, à une géométrie de type affine lorsque *n* est premier. Nous recourrons de nouveau au plan cyclique (*n=5*) que nous compléterons en un premier modèle de géométrie projective d'ordre *5* (en adjoignant les six points idéaux que déterminent les directions ainsi que la droite idéale qu'ils forment):



Par ailleurs, afin de mieux exploiter le caractère de dualité de la géométrie projective, nous avions aussi formé un *"sélecteur de faisceaux"* qui s'est avéré un instrument naturel de calcul en géométries d'ordres *2*, *3* et *4*. La possibilité de réaliser un tel sélecteur et d'*arithmétiser* la géométrie projective d'ordre *5* en produisant un modèle à même l'ensemble des entiers modulo *31*, est liée à l'existence d'une colinéation d'ordre *31* dont le graphe parcourt d'un seul trait tous les points de la géométrie (un peu à la manière d'une courbe de Peano).

Enfin, bien sûr, nous nous doterons d'un *modèle dodécaédrique"* en érigeant sur le polyèdre un réseau approprié de "lignes".

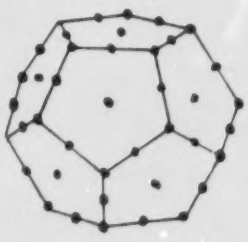
L'exploration consistera à opposer ces trois modèles; mais, pour qu'ils s'enrichissent mutuellement des faits banals que chacun laisse découvrir, il faudra évidemment connecter les modèles entre eux au moyen d'isomorphismes adéquats. C'est ainsi que nous verrons finalement le groupe des *372 000* colinéations de la géométrie projective d'ordre *5* émerger essentiellement de seulement deux transformations banales.

Ayant déjà fourni le modèle cyclique, présentons maintenant les deux autres modèles.

§3. LE DODÉCAÈDRE.–

Que signifie *"ériger une structure sur le dodécaèdre"* ?

Le dodécaèdre possède une gerbe de *31* axes de rotation à laquelle on s'apprête à céder le rôle d'*ensemble* Π *des points* du système géométrique; nous figurerons d'ailleurs cet ensemble par sa trace sur la surface du dodécaèdre.



Ainsi les "lieux" que nous retenons sur le dodécaèdre sont

les *20* sommets, les *30* milieux d'arêtes et les *12* centres des faces; chacun de ces points étant identifié à son antipode. Voilà pour l'"univers des lieux"; mais, pour être élevé au titre de géométrie, l'ensemble Π doit se laisser "stratifier". En effet, si on entreprend par exemple de déplacer un des sommets du dodécaèdre, les autres éléments (sommets, faces, arêtes) sont contraints de "suivre" sous peine de faire éclater le polyèdre; et ce sera parce que l'ensemble Π aura emprunté au dodécaèdre sa "solidité" qu'il distinguera parmi les transformations subies celles qui le respectent. C'est donc en tant qu'objet spatial rigide qu'un certain genre d'activité géométrique se trouve défini par le dodécaèdre (et non en tant qu'ensemble d'éléments amorphe): voir l'ensemble Π à travers la solidité du dodécaèdre c'est lui attribuer une structure particulière et privilégier un groupe d'opérations compatibles avec cette structure, le *groupe des déplacements* (ordre 60).

L'organisation géométrique du dodécaèdre pourrait cependant être décrite autrement que comme objet spatial rigide: l'image du dodécaèdre dans un mirroir, par exemple, est inaccessible par voie de déplacement cependant elle n'en constitue pas moins une déformation pertinente du dodécaèdre. On pourrait donc extraire du dodécaèdre son "caractère isométrique": munir Π d'une structure isométrique consisterait à exprimer que les divers éléments du dodécaèdre sont situés à des écarts bien déterminés les uns par rapport aux autres. Cette interprétation du dodécaèdre privilégierait alors le *groupe des isométries* (ordre 120) comprenant, outre les déplacements, les réflexions.

Sans réclamer toute la liberté qu'accorderait une interprétation topologique, on peut obtenir plus de possibilité d'action que celle consentie par le groupe des 120 isométries en décrivant l'organisation du dodécaèdre simplement en termes d'alignements; la structure projective, en effet, ne permet guère autre chose que de s'*aligner* sur deux points ou de se *localiser* par deux lignes. Il s'agit donc pour nous de distinguer une classe Λ de sous-ensembles de Π pour lui attribuer

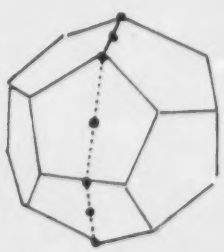
le rôle de *stratification* en lignes; les points et les strates (lignes) étant soumis à l'*axiome d'incidence*:

> *(I). Deux éléments d'une espèce (point ou ligne) déterminent un élément incident commun de l'autre espèce.*

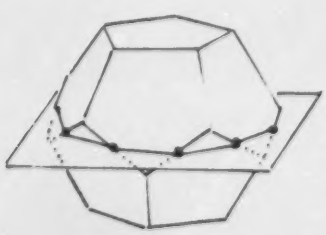
L'action géométrique ainsi privilégiée s'exprime alors en termes de *colinéations* c'est-à-dire de permutations de Π compatibles avec la stratification.

Comment donc s'aligner sur le dodécaèdre? Comment choisir une ligne? Quelle stratification adopter?

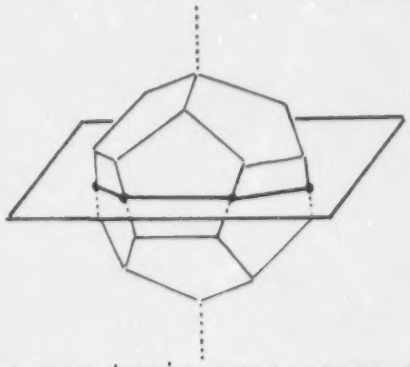
Il est possible d'importer la structure du modèle cyclique en transportant ses *31* lignes (*30* lignes ordinaires et la ligne définie par les points idéaux) au moyen de quelque bijection; il y a là toutefois une prolifération embarrassante de *31!* choix possibles dont certains doivent bien être plus ou moins avantageux que d'autres. Cherchons plutôt parmi les sections du dodécaèdre. Voici, par exemple, une section plane renfermant *6* axes ou couples d'antipodes



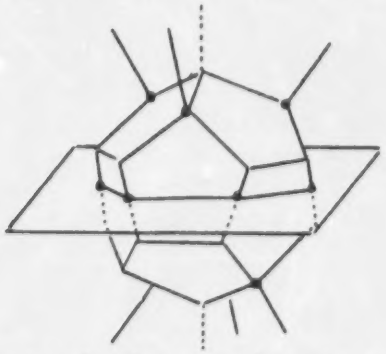
c'est-à-dire le nombre de points requis en géométrie projective d'ordre *5*. Mais toutes les sections ne fournissent pas les *6* éléments escomptés: on ne dénombre que *15* sections "complètes", ce sont celles dont l'axe orthogonal passe par le milieu des arêtes. D'autres sections comprennent *5* axes,



il ne peut être question de les accepter telles quelles. Concédons plutôt qu'"être aligné" pourra signifier autre chose qu'"être coplanaire" et adjoignons l'axe orthogonal aux cinq éléments de la section. Notre récolte de lignes s'accroit ainsi de *6* nouveaux sous-ensembles de Π. Dix lignes restent encore à être trouvées, comment les former? Peut-être en faisant intervenir maintenant les sections perpendiculaires aux axes issus des sommets! Mais ces sections



ne comprennent plus que trois axes; cependant certains triplets d'axes s'offrent spontanément pour les compléter. Ce qui conditionnera notre choix sera naturellement la possibilité de satisfaire à l'axiome d'incidence; c'est ainsi que nous serons contraints de faire le choix suivant



où les trois éléments de la section se voient adjoindre les "trois voisins de l'axe normal".

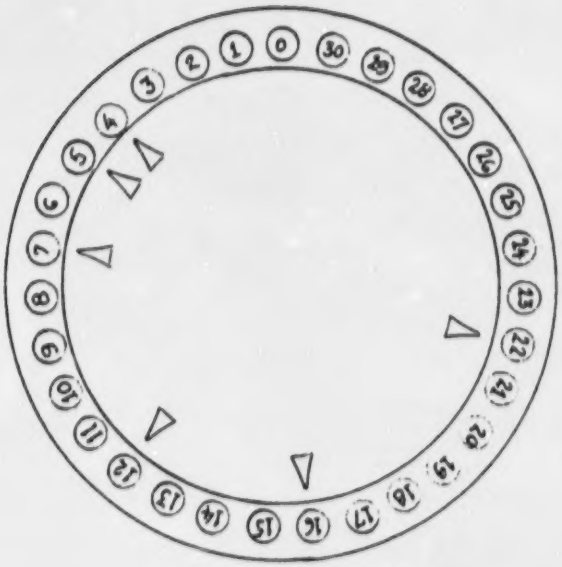
Nous voilà enfin avec une collection Λ de *31* sous-ensembles de Π. Détermine-t-elle une *structure projective?* Est-il vrai que deux points ne se trouveront jamais que dans un et un seul des *31* sous-ensembles? Est-il vrai que deux sous-ensembles de la collection Λ n'admettront jamais qu'un seul élément en commun?

Tout cela n'est guère difficile, ni trop long à vérifier et nous nous retrouvons avec une interprétation du dodécaèdre comme ensemble de lieux Π reliés par un réseau Λ de lignes; ce sera le *modèle dodécaédrique* de la géométrie projective d'ordre *5*.

§4. SÉLECTEUR.–

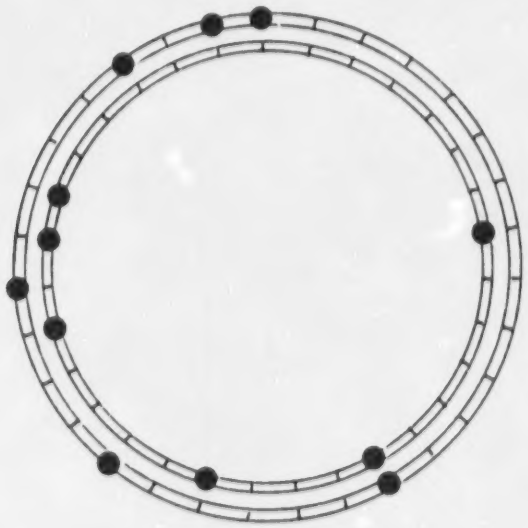
Dans les deux premiers modèles certains éléments semblent bénéficier de "traitement de faveur": tel point se targue d'être le centre d'une face du dodécaèdre ou d'être un sommet, telle ligne s'enorgueillit d'être une section "complète"; dans le modèle cyclique, une des directions se dit "horizontale", une droite et ses points sont qualifiés d'"idéaux". Aucun de ces "droits" pourtant ne semble inscrit dans la "constitution" (*I*); aussi opposerons-nous aux deux modèles précédents un autre modèle plus neutre, plus objectif, sans parasites, ne tolérant aucune discrimination ni entre les points, ni entre les lignes, ni même de l'une des espèces d'éléments à l'égard de l'autre; seule sera mise de l'avant, de façon absolument démocratique, la loi d'incidence (*I*).

Convoquons donc les éléments d'une même espèce (ce pourrait être Π, ce pourrait être Λ) autour d'une "table ronde" et désignons les par les nombres de l'intervalle $[0,31[$. Suivant la "constitution projective (*I*)", chaque ligne $X\in\Lambda$ devient le support d'un ensemble de points $X^*\subset\Pi$ et, de façon duale, chaque point $x\in\Pi$ devient le support d'un ensemble de lignes $x^*\subset\Lambda$; ce sont ces ensembles qu'on appellera *faisceaux* (identifiés par leurs supports) et qu'on se propose de déterminer au moyen d'un *sélecteur* rotatif



muni de *6* flèches pour indiquer les faisceaux. Le défi est naturellement de déceler une répartition adéquate des *6* flèches.

On peut alléger cette recherche et diminuer le tâtonnement ennuyeux en recourant à deux disques concentriques portant des graduations homogènes (de *31* positions chacune)



circulant l'une devant l'autre en transportant une charge de *6* jetons représentant la répartition mise à l'essai. Il s'agit d'obtenir une répartition telle que, sauf pour la position initiale, il n'y aura jamais plus d'une paire de jetons qui se

retrouveront face à face (ce qu'exprime l'axiome (*I*)). C'est ainsi qu'on pourra parvenir à la solution reproduite sur le sélecteur représenté plus haut. Dans ces circonstances, tout faisceau est de la forme

$$X+\mathcal{L} = \{X+l \mid l\in\mathcal{L}\} \quad (mod\ 31)$$

où $\mathcal{L}=\{0,1,3,8,12,18\}$. Et on choisit d'attribuer au support du faisceau $\mathcal{L}$-X l'indice X, alors la relation d'incidence $x\epsilon X$ entre le point d'indice x et la ligne d'indice X s'exprimera

$$x+X \in \mathcal{L} \quad (mod\ 31).$$

Cette relation montre que si $x\epsilon\Pi$ et $X\epsilon\Lambda$ sont incidents, il en ira de même lorsque l'un de ces indices croîtra et que l'autre diminuera d'autant; de sorte que nous pourrions inscrire cette relation fondamentale sur le sélecteur en portant une seconde échelle numérique, semblable à la précédente mais de sens opposé, dissimulée sous le disque rotatoire, et en pratiquant une fenêtre qui ne laissera voir que l'indice désignant le support du faisceau:

La structure géométrique et son caractère d'auto-dualité se trouvent désormais complètement inscrits dans le sélecteur: *l'échelle extérieure représente indifféremment l'ensemble* Π *des points ou l'ensemble* Λ *des lignes, les flèches identifient les faisceaux et la fenêtre livre le support du faisceau.*

§5. JUMELAGE.-

Les modèles que nous venons de proposer apparaissent comme réalisations d'une géométrie qui les transcende, nous avons donc la possibilité d'*altérer notre vision* et d'amplifier la moindre découverte par simple transfert de modèle; encore faut-il, pour cela, avoir établi un "jumelage" entre les modèles. Jumeler signifie coupler les points et les lignes d'un modèle avec ceux d'un autre de manière à ce que les faisceaux se correspondent, autrement dit cela signifie établir un *isomorphisme de structure géométrique*. En particulier brancher le sélecteur sur le dodécaèdre consistera à transporter l'indexation du sélecteur vers le dodécaèdre de manière à ce que le sélecteur fournisse une lecture directe et cohérente de la structure du dodécaèdre.

La construction d'un isomorphisme débute avec le choix d'une partie libre (†). Un point seul est toujours libre; un couple de points distincts saisit une ligne; un triplet de points libre en capte trois mais ne nous apprend rien de plus; un quadruplet libre détermine *6* lignes réparties en *3* couples de côtés dit opposés (parce qu'ils ne se coupent pas sur le quadruplet initial) admettant *3* points "diagonaux" et déterminant les trois côtés de ce qu'on appelle le triangle diagonal. Dans le cas des géométries d'ordre *2* ou *4* l'information cessait de se développer avec ces *7* points et *7* lignes car le triangle diagonal était dégénéré; mais pour la géométrie d'ordre *5*, le processus se poursuit. Procédons donc systématiquement à cette progression d'information associée à la donnée d'un quadruplet libre.

Choisissons d'abord, en géométrie modulo *31*, un premier couple *(0,1)*, puis, pour former un triplet libre, un élément de $\mathcal{C}(0,1)$, soit *2* et, pour former un quadruplet libre, un point hors des côtés du triplet *(0,1,2)*, soit *5*. Le sélecteur permet

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(†) *On dira qu'un ensemble d'éléments de même espèce est libre si aucun élément de l'autre espèce n'est incident à plus de deux de ses éléments. (En géométrie classique, l'ensemble des sommets d'un polygone, l'ensemble des points d'une circonférence sont des ensembles libres). Une suite est dite libre si l'ensemble des éléments de cette suite est libre.*

alors de déterminer les six côtés de ce quadruplet libre Q = (0,1,2,5):

$0^* = \langle 0,1\rangle$ $\quad 1^* = \langle 0,2\rangle$ $\quad 3^* = \langle 0,5\rangle$

$30^* = \langle 1,2\rangle$ $\quad 7^* = \langle 1,5\rangle$ $\quad 29^* = \langle 2,5\rangle$

Ces six côtés se croisent pour donner les trois points diagonaux

$3 = \langle 0^*,29^*\rangle$ $\quad 11 = \langle 1^*,7^*\rangle$ $\quad 9 = \langle 3^*,30^*\rangle$

Ceux-ci livrent à leur tour d'autres lignes, qui redonnent de nouveaux points … produisant une véritable avalanche de points et de lignes où les relations d'interdépendance et de colinéarité doivent être rigoureusement respectées. Voici la suite des éléments progressivement receuillis alternativement dans Π et Λ:

$28^* = \langle 3,11\rangle$
$9^* = \langle 3,9\rangle$
$23^* = \langle 11,9\rangle$

$8 = \langle 0^*,23^*\rangle$
$30 = \langle 1^*,9^*\rangle$
$15 = \langle 3^*,28^*\rangle$
$4 = \langle 30^*,28^*\rangle$
$25 = \langle 7^*,9^*\rangle$
$20 = \langle 29^*,23^*\rangle$

$18^* = \langle 0,25\rangle$
$12^* = \langle 0,20\rangle$
$2^* = \langle 1,30\rangle$
$17^* = \langle 1,15\rangle$
$11^* = \langle 1,20\rangle$
$10^* = \langle 2,8\rangle$
$16^* = \langle 2,15\rangle$
$6^* = \langle 2,25\rangle$
$26^* = \langle 5,8\rangle$
$13^* = \langle 5,30\rangle$
$27^* = \langle 5,4\rangle$
$24^* = \langle 8,15\rangle$
$19^* = \langle 30,15\rangle$
$14^* = \langle 4,25\rangle$

$18 = \langle 0^*,16^*\rangle$
$12 = \langle 0^*,6^*\rangle$
$17 = \langle 1^*,17^*\rangle$
$7 = \langle 1^*,11^*\rangle$
$29 = \langle 3^*,2^*\rangle$
$28 = \langle 3^*,11^*\rangle$
$13 = \langle 30^*,18^*\rangle$
$19 = \langle 30^*,12^*\rangle$
$24 = \langle 7^*,8^*\rangle$
$27 = \langle 7^*,12^*\rangle$
$10 = \langle 29^*,8^*\rangle$
$14 = \langle 29^*,18^*\rangle$
$21 = \langle 28^*,18^*\rangle$
$6 = \langle 28^*,12^*\rangle$
$23 = \langle 9^*,8^*\rangle$
$22 = \langle 9^*,12^*\rangle$
$26 = \langle 23^*,8^*\rangle$
$16 = \langle 23^*,18^*\rangle$

$18^* = \langle 0,13\rangle$
$15^* = \langle 3,17\rangle$
$5^* = \langle 3,7\rangle$
$21^* = \langle 11,18\rangle$
$20^* = \langle 11,12\rangle$
$25^* = \langle 9,18\rangle$
$22^* = \langle 9,12\rangle$

Et voilà donc que les *62* éléments de la géométrie modulo *31* se trouvent tous déterminés par la seule donnée du quadruplet libre initial Q = *(0,1,2,5)*. Naturellement le même *schéma de prospection* des éléments peut être repris dans le modèle dodécaédrique ou dans le modèle du plan cyclique et ainsi donner lieu à un *isomorphisme géométrique* par lequel tout point et toute ligne se verra attribuer un indice

$$n \longmapsto a_n$$
$$n^* \longmapsto A_n$$

respectant les relations essentielles incorporées dans le sélecteur. En particulier, on pourra construire un isomorphisme de la géométrie projective (Π,Λ) sur elle-même ou sur sa duale, cela même que nous appelons *colinéation* ou *corrélation*, à partir de ses valeurs sur un quadruplet libre quelconque; en conséquence

$$Card\ \mathcal{C} = 31\times30\times25\times16 = 372{,}000.$$

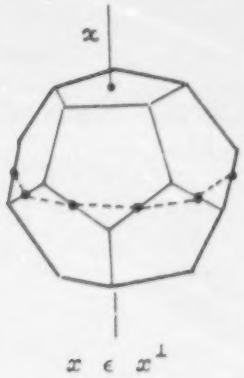
§6. INDEXATION DU DODÉCAÈDRE ET DU PLAN CYCLIQUE.–

Ce que nous apprend le résultat précédent c'est qu'il y a plus d'un tiers de million de façons possibles d'indexer les points du dodécaèdre. Il serait donc présomptueux de procéder arbitrairement. Pour obtenir un choix prometteur, souvenons-nous que le caractère d'auto-dualité de la géométrie projective est primordial; or dans le cas du sélecteur, cette "symétrie logique interne" se traduit concrètement par la présence d'une corrélation explicite, celle associant à tout point la ligne de même indice. D'autre part, on relève aussi en géométrie du dodécaèdre une corrélation analogue, naturellement inscrite dans le modèle, savoir celle qui associe à tout axe $x\in\Pi$ la ligne $x^{\perp}\in\Lambda$ issue de la section orthogonale.

Est-il possible d'indexer les axes du dodécaèdre de manière que la corrélation naturelle que porte le sélecteur se traduise par la correspondance $\perp$ dans le dodécaèdre?

La réalisation de cet objectif signifierait que la relation d'incidence telle qu'inscrite sur le sélecteur aurait été traduite en la relation d'orthogonalité. La ligne provenant d'une

section donnée du dodécaèdre porterait alors le même indice que l'axe normal; mais, on l'a vu, six lignes *comprennent* leur normale

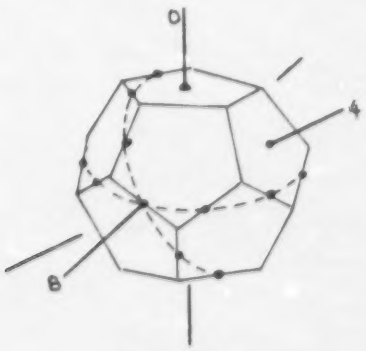


et de ce fait porteront même indice qu'un de leurs éléments. Le sélecteur nous révèle aussitôt que la relation $x \in x^*$ possède également six solutions dans $[0,31[$:

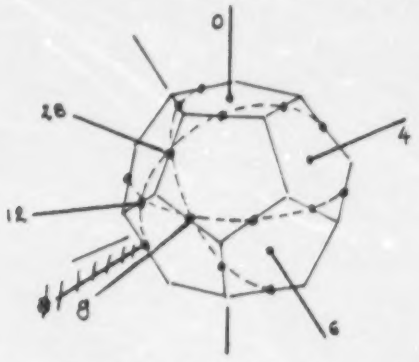
$$0,\ 4,\ 6,\ 9,\ 16,\ 17.$$

En conséquence, les indices *0, 4, 6, 9, 16, 17* devront être réservés aux six axes passant par le centre des faces du dodécaèdre.

Puisque cet ensemble d'indices est libre, de même que l'ensemble des axes issus du centre des faces, une solution de notre problème d'indexation du dodécaèdre pourrait s'obtenir en construisant l'isomorphisme déterminé par l'application d'un quadruplet extrait de $\{0,4,6,9,16,17\}$, par exemple $(0,4,6,9)$, sur les centres de quatre faces convenablement choisies. Les deux premiers indices *0* et *4* seront attribués aux centres de deux faces voisines quelconques (compte tenu de l'identification des antipodes, deux faces quelconques se retrouvent nécessairement adjacentes). Mais déjà certains liens s'établissent:



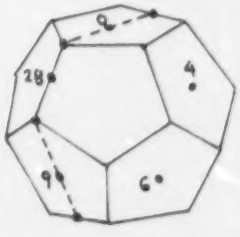
les lignes du docécaèdre orthogonales aux axes *0* et *4* s'interceptent en un axe qui devra recevoir, toujours selon le sélecteur, l'indice *8*: $8 = 0^* \cap 4^*$. Le troisième point *6* semble encore pouvoir s'appliquer au centre de n'importe quelle autre face



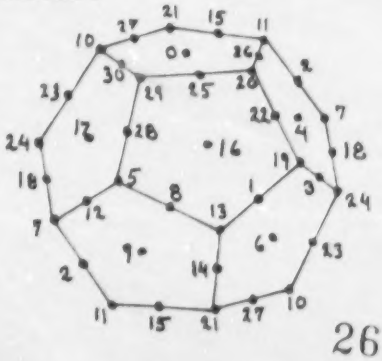
mais il détermine alors l'attribution des indices $12 = 0^* \cap 6^*$ et $28 = 4^* \cap 6^*$. Trois faces demeurent disponibles pour recevoir le dernier élément du quadruplet $(0,4,6,9)$ mais ce serait une erreur de l'attribuer arbitrairement puisqu'il doit être aligné avec *0* et *28*:

$$9 \in (0,28) = \{0,5,9,15,28,29\},$$

ce qui ne laisse plus aucune liberté quant au choix du point *9*.

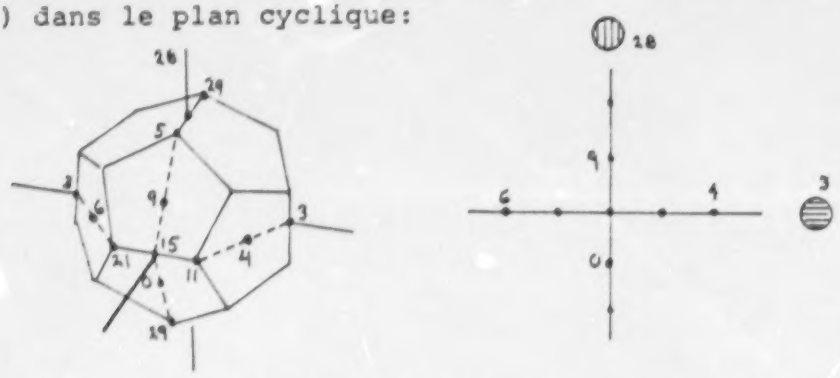


Ce sera donc cette correspondance particulière entre le quadruplet $(0,4,6,9)$ et quatre faces du dodécaèdre qui, prolongées suivant le processus de jumelage exposé au numéro précédent, fournira l'indexation finale



qui s'avère être une solution correcte à notre problème.

Enfin, pour compléter le jumelage de nos trois modèles, nous nous contenterons simplement d'emprunter au dodécaèdre l'indexation résultant de l'application suivante du quadruplet *(0,4,6,9)* dans le plan cyclique:



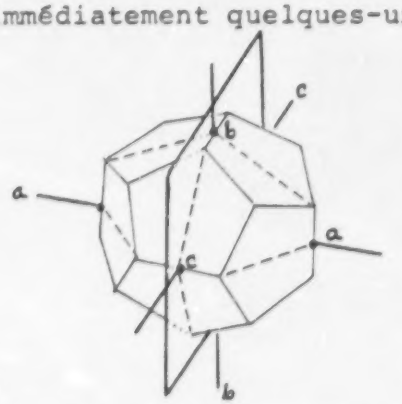
Cette indexation est destinée à coupler le trièdre d'axes orthogonaux *(3,28,15)* avec un triplet de points bien spéciaux du plan cyclique:

§7. LA GÉNÉRATION DU GROUPE DES COLINÉATIONS.-

Comme nous le rappelions plus tôt, le dodécaèdre, en nous présentant les rotations autour de ses axes, permettait déjà d'engendrer les *30* déplacements possibles; en exploitant les réflexions dans ses divers plans de symétrie, il conduisait au groupe des *60* isométries; mais nous savons qu'il y a beaucoup plus et qu'il existe, pour soutenir l'action, au-delà d'un tiers de million de transformations compatibles avec la structure.

Comment harnacher ce flot de transformations en ne consentant qu'une capitalisation d'effort minimale? Naturellement, ce pourrait être en les exprimant toutes par certaines d'entre elles. Nous découvrirons que deux colinéations seulement suffisent, en un sens, pour les engendrer toutes.

Le groupe des colinéations, opérant transitivement dans l'ensemble des quadruplets libres, opérera transitivement a fortiori dans l'ensemble des triplets libres, de sorte que deux quadruplets libres peuvent toujours se rencontrer dans l'"île des quadruplets libres prolongeant un triplet donné", c'est-à-dire dans l'ensemble des quadruplets libres de la forme (a,b,c,x) où x doit être choisi hors de l'ensemble $\langle a,b,c\rangle$ des points situés sur les côtés du triangle de sommets a, b, c. Mais, à l'intérieur même de cette île, pour espérer transformer l'un en l'autre deux quadruplets, il faudra recourir à des colinéations laissant chacun des points a, b, c invariants. Le dodécaèdre nous en offre immédiatement quelques-unes:

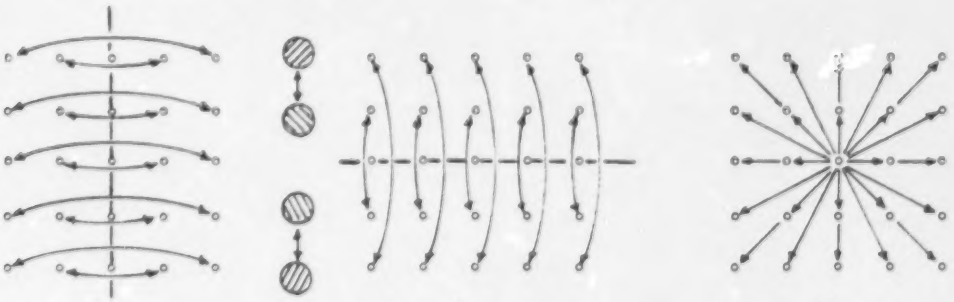


les réflexions $\mathcal{R}_a$, $\mathcal{R}_b$, $\mathcal{R}_c$ relatives à un trièdre (a,b,c) d'axes orthogonaux du dodécaèdre, par exemple, laissent toute trois a, b, c invariants. Toutefois l'action conjointe de ces trois réflexions ne suffit guère à opérer transitivement dans tout l'ensemble $\mathcal{C}\langle a,b,c\rangle$ où doit évoluer le quatrième élément des quadruplets de l'île. Cela n'est pas surprenant puisque les réflexions, en tant qu'isométries, ne savent guère transformer un sommet du dodécaèdre en le centre d'une face par exemple.

On pourrait rénover notre perception des réflexions en les exportant vers un autre modèle; ainsi la réflexion $\mathcal{R}_1$ par exemple prend, dans le modèle du disque, la forme suivante:

$\alpha = \mathcal{R}_1$

qui resurgira plus loin. Dans le plan cyclique, $\mathcal{R}_1$ prendra aussi une forme relativement déconcertante, cependant $\mathcal{R}_3$, $\mathcal{R}_{28}$ et $\mathcal{R}_{15}$ deviendront des transformations familières:



$\mathcal{R}_3$ $\mathcal{R}_{28}$ $\mathcal{R}_{15}$

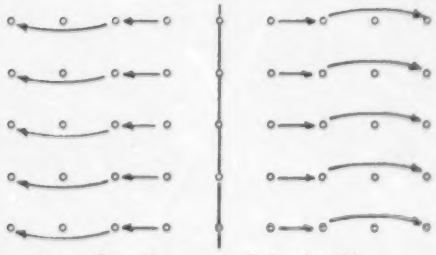
(sans doute en raison du fait que le triplet $(a,b,c)=(3,28,15)$ est le seul des triplets d'axes orthogonaux du dodécaèdre qui engage la droite idéale $(3,28)$ du modèle cyclique).

Toutes ces colinéations peuvent être subsumées sous le concept d'homologie:

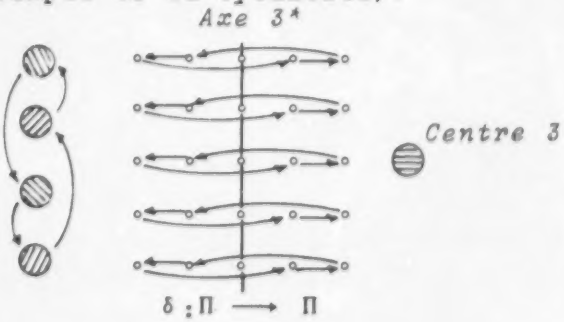
Définition.— Une colinéation pour laquelle il existe un point (le centre) et une ligne (l'axe) non incidents supportant chacun un faisceau d'éléments invariants s'appelle une homologie.

Une homologie de centre et d'axe donnés est complètement caractérisée par la donnée d'une seule "flèche" supplémentaire, ce qui permet de s'en construire facilement. Formons, par exemple, l'homologie δ de centre *3*, d'axe $3^*=(15,28)$, portant le point *22* en *30* (points que $\mathcal{R}_3$, $\mathcal{R}_{28}$, $\mathcal{R}_{15}$ n'arrivaient pas à mettre en communication). Un simple prolongement par colinéation livrera progressivement

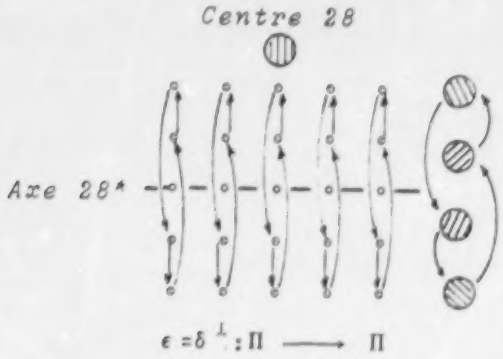
puis



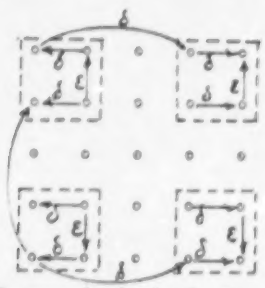
et enfin (tenant compte de la cyclicité):



Cette pseudo-réflexion δ ne relie pas encore tous les points de $\mathcal{C}\langle 3,28,15\rangle$ mais il suffit de la "retourner"



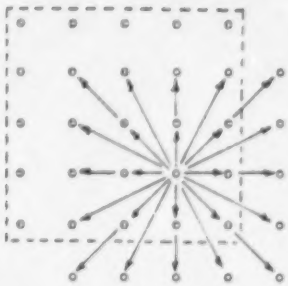
pour découvrir enfin que, par leur action conjointe, les colinéations δ et ϵ garantissent la transitivité dans l'ensemble des quadruplets libres de l'île. Voici en effet un réseau constitué de flèches empruntées aux graphes de δ et ϵ et reliant entre eux tous les éléments de l'ensemble $\mathcal{C}\langle 3,28,15\rangle$ des points x susceptibles de produire un quadruplet libre $(3,28,15,x)$.



Transitivité dans l'île $\{(3,28,15,x)\}$

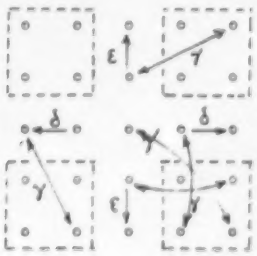
Pour garantir maintenant la transitivité dans l'"univers" des quadruplets libres (et non seulement ceux de l'île) il faudra savoir reconduire tout quadruplet dans l'île c'est-à-dire pouvoir transformer tout triplet libre en la "base" $(3,28,15)$. Cela aussi on pourra l'entreprendre en deux étapes: d'abord reconduire les triplets dans l'"île des triplets libres issus du couple $(3,28)$", $\{(3,28,x) \mid x \in \mathcal{C}\langle 3,28\rangle\}$, puis assurer la transitivité dans l'île des triplets libres elle-même. Or on arrive à faire la liaison entre tous les points de $\mathcal{C}\langle 3,28\rangle$ (c'est-à-dire tous les

points ordinaires du plan cyclique) au moyen de colinéations respectant *3* et *28* en joignant à δ et ϵ l'action de la symétrie γ de centre *18*,



$\gamma : \Pi \longrightarrow \Pi$ *(Centre 18)*

(qu'il n'aurait guère été facile de déceler dans le modèle dodécaédrique). Voici, en effet, comment γ rattache les nouveaux points au bloc de ceux qui sont déjà reliés par δ et ϵ :



Ainsi est assurée la transitivité dans l'île des triplets libres issus de *(3,28)*.

Maintenant pour pouvoir acheminer tous les triplets libres vers l'île, il nous faut savoir propulser tous les couples de points distincts sur la cible *(3,28)*. Utilisant toujours la même stratégie, essayons d'abord d'établir la transitivité dans l'île des couples issus du point *3*, $\{(3,x) \mid x \neq 3\}$. Déjà les colinéations γ, δ, ϵ opèrent dans cette île (puisqu'elles laissent *3* invariant) mais elles n'arrivent pas encore à interchanger tous les éléments de l'ensemble $\Pi - \{3\}$ et il nous faudra chercher de nouvelles colinéations respectant le point *3*.

Si on transporte la symétrie centrale γ dans le modèle arithmétique du disque, on a la surprise de découvrir qu'à rotation du disque près, le graphe de γ coïncide avec celui de $\mathcal{R}_1$. Nous

voyons ainsi malgré nous l'action sur les "atomes" (points) se répercuter au niveau de cette "molécule" que constitue la réflexion $\mathcal{R}_1$:

$$\gamma = \rho^{17}(\mathcal{R}_1)$$

(ρ désignant la rotation du disque $x \longmapsto x+1 \pmod{31}$). Et c'est maintenant l'orbite entier de l'objet $\mathcal{R}_1$ (sous l'effet des rotations du disque) que nous avons à notre disposition. Il n'en faudra pas davantage pour pouvoir terminer notre projet. En effet, l'adjonction à γ, δ, ϵ de

$$\beta = \rho(\mathcal{R}_1)$$

permet maintenant d'opérer transitivement dans $\Pi-\{3\}$ (donc aussi dans l'île des couples de la forme $(3,x)$, $x \neq 3$) et, enfin, la réflexion

$$\alpha = \mathcal{R}_1$$

elle-même permet à tout couple de prendre la forme $(3,x)$ c'est-à-dire de parvenir dans l'île.

§8. QUANTA D'ACTION.–

Puisqu'en géométrie d'ordre 5 une colinéation est caractérisée par son effet sur un quadruplet libre, c'est toute l'activité géométrique (représentée par le groupe $\mathcal{C}$ des colinéations) qui est contenue en germes dans la réflexion $\alpha = \mathcal{R}_1$ du dodécaèdre et la pseudo-réflexion δ du plan cyclique. En effet α, β, γ ne sont que des lectures d'un même graphe vu en trois orientations sur le disque

$$\alpha = \mathcal{R}_1, \quad \beta = \rho(\mathcal{R}_1), \quad \gamma = \rho^{17}(\mathcal{R}_1)$$

tandis que δ et ϵ dérivent l'une de l'autre par retournement dans le plan cyclique

$$\epsilon = \delta^{\perp}.$$

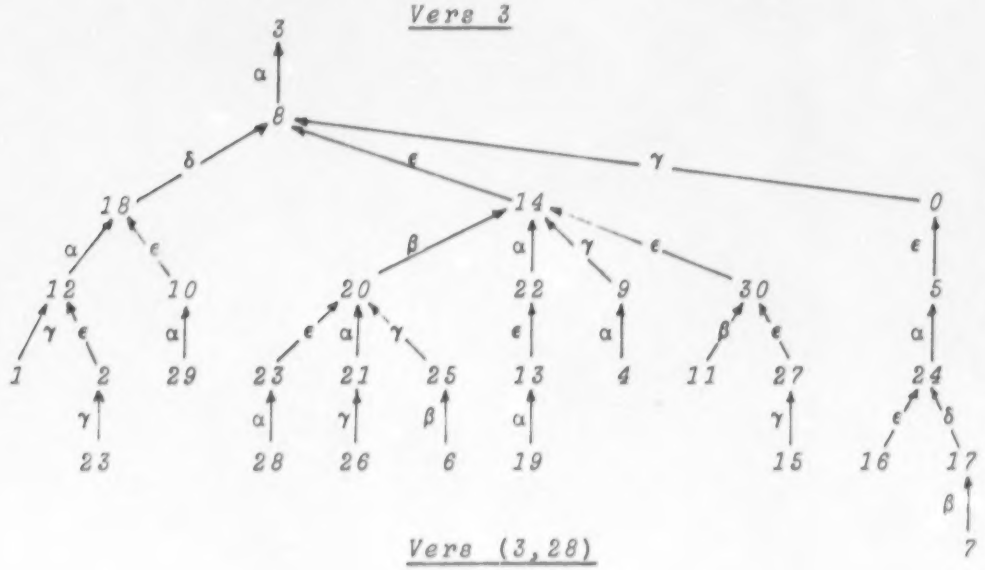
La réflexion $\mathcal{R}_1$ et la pseudo-réflexion δ apparaissent donc en quelque sorte comme "quanta d'action" susceptibles de regénérer la classe $\mathcal{C}$ de toutes les "actions géométriques".

§9. ILLUSTRATIONS.-

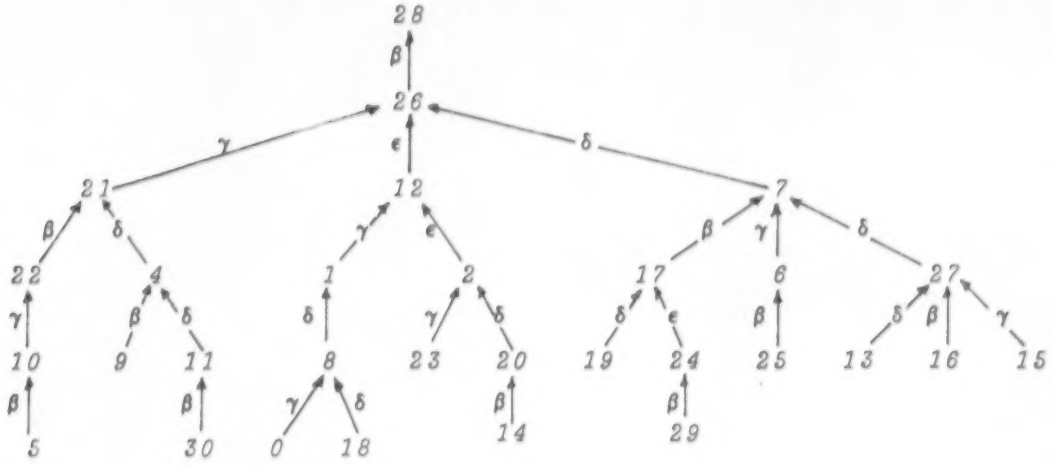
A titre d'illustration indiquons comment peut s'exprimer une colinéation quelconque $f \in \mathcal{C}$ en termes de quanta.

Puisque toute colinéation f est déterminée par ses valeurs sur un quadruplet libre quelconque et en particulier sur le quadruplet $f^{-1}(3,28,15,1)$ (car nous privilégierons la cible $(3,28,15,1)$), nous nous doterons de "cartes de voyage" indiquant les itinéraires vers 3, vers $(3,28)$, vers $(3,28,15)$ et finalement vers la destination finale $(3,28,15,1)$.

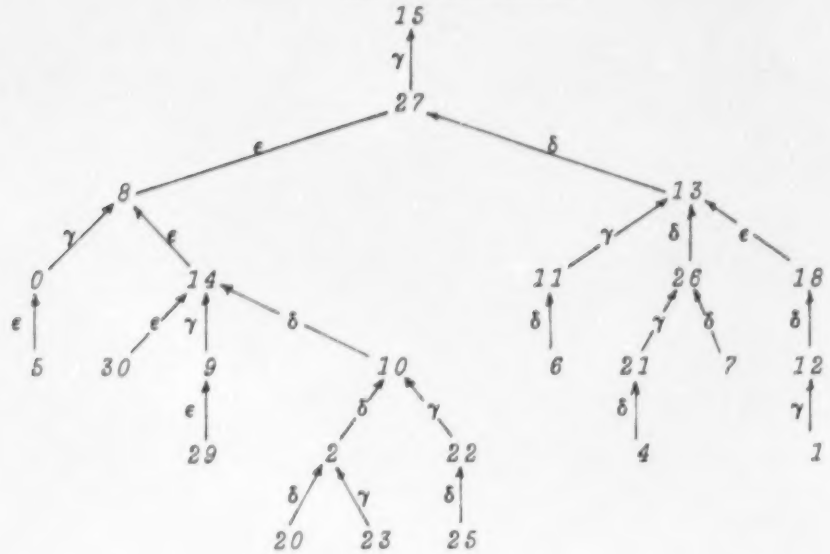
Vers 3



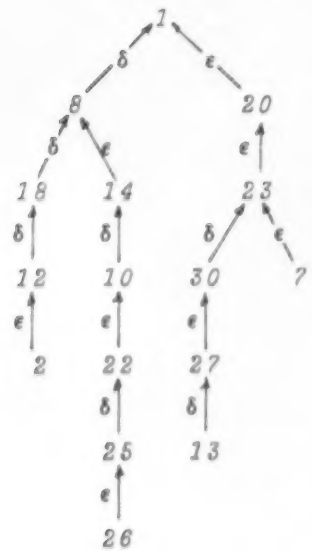
Vers (3,28)



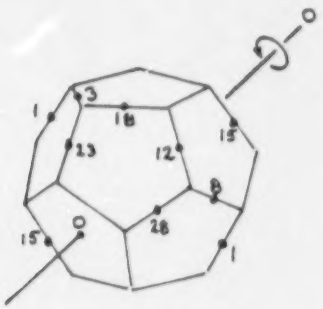
Vere (3,28,15)



Vere (3,28,15,1)



Rotation du dodécaèdre autour de l'axe 0.-



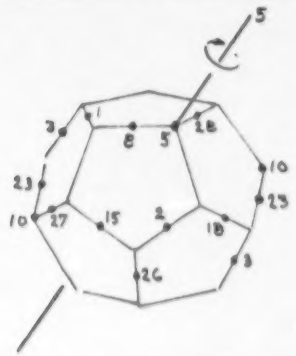
Une telle rotation f peut être caractérisée par le fait de transformer $(8,23,26,12)$ en la cible $(3,28,15,1)$; ce qu'on réalise progressivement comme suit, d'après les cartes de voyage:

$$(8,23,26,12) \xrightarrow{\alpha} (3,28,15,18) \xrightarrow{\delta\circ\delta} (3,28,15,1)$$

Ainsi

$$f = \delta\circ\delta\circ\alpha.$$

Rotation du dodécaèdre autour du sommet 5.-



Une telle rotation g peut être caractérisée comme prolongement de la correspondance

$$(26,8,23,2) \longrightarrow (3,28,15,1)$$

que l'on peut réaliser progressivement, suivant les cartes, de la façon suivante:

$$(26,8,23,2) \xrightarrow{\alpha\circ\epsilon\circ\beta\circ\alpha\circ\gamma} (3,10,18,27) \xrightarrow{\beta\circ\gamma\circ\beta\circ\gamma} (3,28,18,2)$$

$$\xrightarrow{\gamma\circ\delta\circ\epsilon} (3,28,15,18) \xrightarrow{\delta\circ\delta} (3,28,15,1)$$

Ainsi

$$g = \delta^2\circ\gamma\circ\delta\circ\epsilon\circ\beta\circ\gamma\circ\beta\circ\gamma\circ\alpha\circ\epsilon\circ\beta\circ\alpha\circ\gamma$$

Rotation du sélecteur.–

Cette remarquable colinéation d'ordre 31

$$\rho : x \longmapsto x+1 \pmod{31}$$

est caractérisée par la correspondance

$$(2,27,14,0) \longrightarrow (3,28,15,1)$$

qu'on réalise progressivement comme suit:

$$(2,27,14,0) \xrightarrow{\alpha\circ\delta\circ\alpha\circ\varepsilon} (3,28,8,22) \xrightarrow{\gamma\circ\varepsilon}$$

$$(3,28,15,22) \xrightarrow{\delta\circ\varepsilon\circ\delta\circ\varepsilon} (3,28,15,1).$$

En conséquence

$$\rho = \delta\circ\varepsilon\circ\delta\circ\varepsilon\circ\gamma\circ\varepsilon\circ\alpha\circ\delta\circ\alpha\circ\varepsilon .$$

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